IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

Mathematics
Higher level

Specimen questions paper 1 and paper 2

For first examinations in 2008

CONTENTS

Introduction

The assessment model has been changed for May 2008:

- Paper 1 and paper 2 will both consist of section A, short questions answered on the paper (similar to the current paper 1), and section B, extended-response questions answered on answer sheets (similar to the current paper 2).

- Calculators will not be allowed on paper 1.

- Graphic display calculators (GDCs) will be required on paper 2 and paper 3.

The revised assessment model for external components will be:

Paper 1 (no calculator allowed) 30% 2 hrs

Section A 60 marks
Compulsory short-response questions based on the compulsory core of the syllabus.

Section B 60 marks
Compulsory extended-response questions based on the compulsory core of the syllabus.

Paper 2 (calculator required) 30% 2 hrs

Section A 60 marks
Compulsory short-response questions based on the compulsory core of the syllabus.

Section B 60 marks
Compulsory extended-response questions based on the compulsory core of the syllabus.

Paper 3 (calculator required) 20% 1 hr

Extended-response questions based mainly on the syllabus options. 60 marks

Full details can be found in the second edition of the mathematics HL guide which was sent to schools in September 2006 and is available on the online curriculum centre (OCC).

Why are these changes being made?

Experience has shown that certain papers can be answered using the GDC very little, although some students will answer the same papers by using a GDC on almost every question. We have seen some very interesting and innovative approaches used by students and teachers, however there have been occasions when the paper setters wished to assess a particular skill or approach. The fact that candidates had a GDC often meant that it was difficult (if not impossible) to do this. The problem was exacerbated by the variety of GDCs used by students worldwide. The examining team feel that a calculator-free environment is needed in order to better assess certain knowledge and skills.

How will these changes affect the way the course is taught?

Most teachers should not find it necessary to change their teaching in order to be able to comply with the change in the assessment structure. Rather it will give them the freedom to emphasize the analytical approach to certain areas of the course that they may have been neglecting somewhat, not because they did not deem it relevant or even essential, but because it was becoming clear that technology was "taking the upper hand" and ruling out the need to acquire certain skills.

Are there changes to the syllabus content?

No, it should be emphasized that it is only the assessment model that is being changed. There is no intention to change the syllabus content. Neither is there any intention to reduce the role of the GDC, either in teaching or in the examination.

Any references in the subject guide to the use of a GDC will still be valid, for example, finding the inverse of a 3x3 matrix using a GDC; this means that these will not appear on paper 1. Another example of questions that will not appear on paper 1 is statistics questions requiring the use of tables. In trigonometry, candidates are expected to be familiar with the characteristics of the sin, cos and tan curves, their symmetry and periodic properties, including knowledge of the ratios of $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 180^\circ$ and how to derive the ratios of multiples by using the symmetry of the curves, for example, $\sin 210^\circ = -\sin 30^\circ$.

What types of questions will be asked on paper 1?

Paper 1 questions will mainly involve analytical approaches to solutions rather than requiring the use of a GDC. It is not intended to have complicated calculations with the potential for careless errors. However, questions will include some arithmetical manipulations when they are essential to the development of the question.

What types of questions will be asked on paper 2?

These questions will be similar to those asked on the current papers. Students must have access to a GDC at all times, however not all questions will necessarily require the use of the GDC. There will be questions where a GDC is not needed and others where its use is optional. There will be some questions that cannot be answered without a GDC that meets the minimum requirements.

What is the purpose of this document?

This document is a combination of the original specimen papers for papers 1 and 2 (published in November 2004) and the new specimen questions for paper 1 (published online in November 2006). It should be noted that this is not two specimen papers but a collection of questions illustrating the types of questions that may be asked on each paper. Thus they will not necessarily reflect balanced syllabus coverage, nor the relative importance of the syllabus topics.

In order to provide teachers with information about the examinations, the rubrics for each paper and section are included below. In papers 1 and 2 Section A questions should be answered in the spaces provided, and Section B questions on the answer sheets provided by the IBO. Graph paper should be used if required. The answer spaces have been included with the first 2 questions of Section A on each paper. Paper 3 has not changed.

Paper 1

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer all the questions in the spaces provided. Working may be continued below the lines, if necessary.

Section B

Answer all the questions on the answer sheets provided. Please start each question on a new page.

Paper 2

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer all the questions in the spaces provided. Working may be continued below the lines, if necessary.

Section B

Answer all the questions on the answer sheets provided. Please start each question on a new page.

Paper 3

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Markscheme instructions

A. Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

B. Using the markscheme

Follow through (***FT***) marks: Only award ***FT*** marks when a candidate uses an incorrect answer in a subsequent **part**. Any exceptions to this will be noted on the markscheme. Follow through marks are now the exception rather than the rule within a question or part question. Follow through marks may only be awarded to work that is seen. Do **not** award ***N FT*** marks. If the question becomes much simpler then use discretion to award fewer marks. If a candidate mis-reads data from the **question** apply follow-through.

Discretionary (d) marks: There will be rare occasions where the markscheme does not cover the work seen. In such cases, **(d)** should used to indicate where an examiner has used discretion. It must be accompanied by a brief note to explain the decision made.

It is important to understand the difference between "**implied**" marks, as indicated by the brackets, and marks which can only be awarded for work seen - no brackets. The implied marks can only be awarded if **correct** work is seen or implied in subsequent working. Normally this would be in the next line.

Where ***M1 A1*** are awarded on the same line, this usually means ***M1*** for an attempt to use an appropriate formula, ***A1*** for correct substitution.

As ***A*** marks are normally **dependent** on the preceding ***M*** mark being awarded, it is not possible to award ***M0 A1***.

As ***N*** marks are only awarded when there is no working, it is not possible to award a mixture of ***N*** and other marks.

Accept all correct alternative methods, even if not specified in the markscheme Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative (part) solutions, are indicated by **EITHER**….**OR**. Where possible, alignment will also be used to assist examiners to identify where these alternatives start and finish.

Unless the question specifies otherwise, accept **equivalent forms**. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer The markscheme indicate the required answer, by allocating full marks at that point. Once the correct answer is seen, ignore further working, unless it contradicts the answer.

Brackets will also be used for what could be described as the well-expressed answer, but which candidates may not write in examinations. Examiners need to be aware that the marks for answers should be awarded for the form preceding the brackets *e.g.* in differentiating $f(x) = 2\sin(5x-3)$, the markscheme says

$$f'(x) = \big(2\cos(5x-3)\big)5 \quad \big(=10\cos(5x-3)\big) \qquad \textbf{\textit{A1}}$$

This means that the ***A1*** is awarded for seeing $\big(2\cos(5x-3)\big)5$, although we would normally write the answer as $10\cos(5x-3)$.

As this is an international examination, all **alternative forms of notation** should be accepted.

Where the markscheme specifies ***M2***, ***A3***, *etc.*, for an answer do NOT split the marks unless otherwise instructed.

Do **not** award full marks for a correct answer, all working must be checked.

Candidates should be penalized **once IN THE PAPER** for an accuracy error (***AP***). There are two types of accuracy error:

- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule is *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figure*s.

Paper 1

Section A questions

1. *[Maximum mark: 5]*

Given that $4\ln 2 - 3\ln 4 = -\ln k$, find the value of k.

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2. *[Maximum mark: 5]*

Solve the equation $\log_3(x+17)-2=\log_3 2x$.

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3. *[Maximum mark: 6]*

Solve the equation $2^{2x+2} - 10 \times 2^{x} + 4 = 0$, $x \in \mathbb{R}$.

4. *[Maximum mark: 7]*

Given that $(a + b\text{i})^2 = 3 + 4\text{i}$ obtain a pair of simultaneous equations involving *a* and *b*. Hence find the two square roots of $3 + 4\text{i}$.

5. *[Maximum mark: 5]*

Given that $2 + \text{i}$ is a root of the equation $x^3 - 6x^2 + 13x - 10 = 0$ find the other two roots.

6. *[Maximum mark: 7]*

Given that $|z| = \sqrt{10}$, solve the equation $5z + \dfrac{10}{z^*} = 6 - 18\text{i}$, where z^* is the conjugate of *z*.

7. *[Maximum mark: 8]*

Find the three cube roots of the complex number 8i. Give your answers in the form $x + \text{i}y$.

8. *[Maximum mark: 9]*

Solve the simultaneous equations

$$\text{i}z_1 + 2z_2 = 3$$
$$z_1 + (1 - \text{i})\,z_2 = 4$$

giving z_1 and z_2 in the form $x + \text{i}y$, where *x* and *y* are real.

9. *[Maximum mark: 6]*

Find *b* where $\dfrac{2 + b\text{i}}{1 - b\text{i}} = -\dfrac{7}{10} + \dfrac{9}{10}\text{i}$.

10. *[Maximum mark: 6]*

Given that $z = (b + \mathrm{i})^2$, where b is real and positive, find the value of b when $\arg z = 60^\circ$.

11. *[Maximum mark: 5]*

Find all values of x that satisfy the inequality $\dfrac{2x}{|x-1|} < 1$.

12. *[Maximum mark: 6]*

The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $(x-2)$ as when divided by $(x+1)$. Find the value of a.

13. *[Maximum mark: 6]*

The functions f and g are defined by $f : x \mapsto \mathrm{e}^x$, $g : x \mapsto x + 2$.

Calculate

(a) $f^{-1}(3) \times g^{-1}(3)$; *[3 marks]*

(b) $(f \circ g)^{-1}(3)$. *[3 marks]*

14. *[Maximum mark: 6]*

Solve $\sin 2x = \sqrt{2} \cos x$, $0 \le x \le \pi$.

15. *[Maximum mark: 6]*

The obtuse angle B is such that $\tan B = -\dfrac{5}{12}$. Find the values of

(a) $\sin B$; *[1 mark]*

(b) $\cos B$; *[1 mark]*

(c) $\sin 2B$; *[2 marks]*

(d) $\cos 2B$. *[2 marks]*

16. *[Maximum mark: 5]*

Given that $\tan 2\theta = \frac{3}{4}$, find the possible values of $\tan\theta$.

17. *[Maximum mark: 9]*

Let $\sin x = s$.

(a) Show that the equation $4\cos 2x + 3\sin x \operatorname{cosec}^3 x + 6 = 0$ can be expressed as $8s^4 - 10s^2 + 3 = 0$. *[3 marks]*

(b) Hence solve the equation for x, in the interval $[0, \pi]$. *[6 marks]*

18. *[Maximum mark: 9]*

(a) If $\sin(x - \alpha) = k \sin(x + \alpha)$ express $\tan x$ in terms of k and α. *[3 marks]*

(b) Hence find the values of x between 0° and 360° when $k = \frac{1}{2}$ and $\alpha = 210^\circ$. *[6 marks]*

19. *[Maximum mark: 6]*

The angle θ satisfies the equation $2\tan^2\theta - 5\sec\theta - 10 = 0$, where θ is in the second quadrant. Find the value of $\sec\theta$.

20. *[Maximum mark: 5]*

Find the determinant of $\boldsymbol{A}$, where $\boldsymbol{A} = \begin{pmatrix} 3 & 1 & 2 \\ 9 & 5 & 8 \\ 7 & 4 & 6 \end{pmatrix}$.

21. *[Maximum mark: 5]*

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ and $\boldsymbol{A}^2$ is a matrix whose entries are all 0, find k.

22. *[Maximum mark: 5]*

Given that $\boldsymbol{M} = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ and that $\boldsymbol{M}^2 - 6\boldsymbol{M} + k\boldsymbol{I} = 0$ find k.

23. *[Maximum mark: 6]*

The square matrix $\boldsymbol{X}$ is such that $\boldsymbol{X}^3 = 0$. Show that the inverse of the matrix $\boldsymbol{(I - X)}$ is $\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$.

24. *[Maximum mark: 6]*

The line L is given by the parametric equations $x = 1 - \lambda$, $y = 2 - 3\lambda$, $z = 2$. Find the coordinates of the point on L which is nearest to the origin.

25. *[Maximum mark: 5]*

Flowering plants are randomly distributed around a field according to a Poisson distribution with mean μ. Students find that they are twice as likely to find exactly ten flowering plants as to find exactly nine flowering plants in a square metre of field. Calculate the expected number of flowering plants in a square metre of field.

26. *[Maximum mark: 6]*

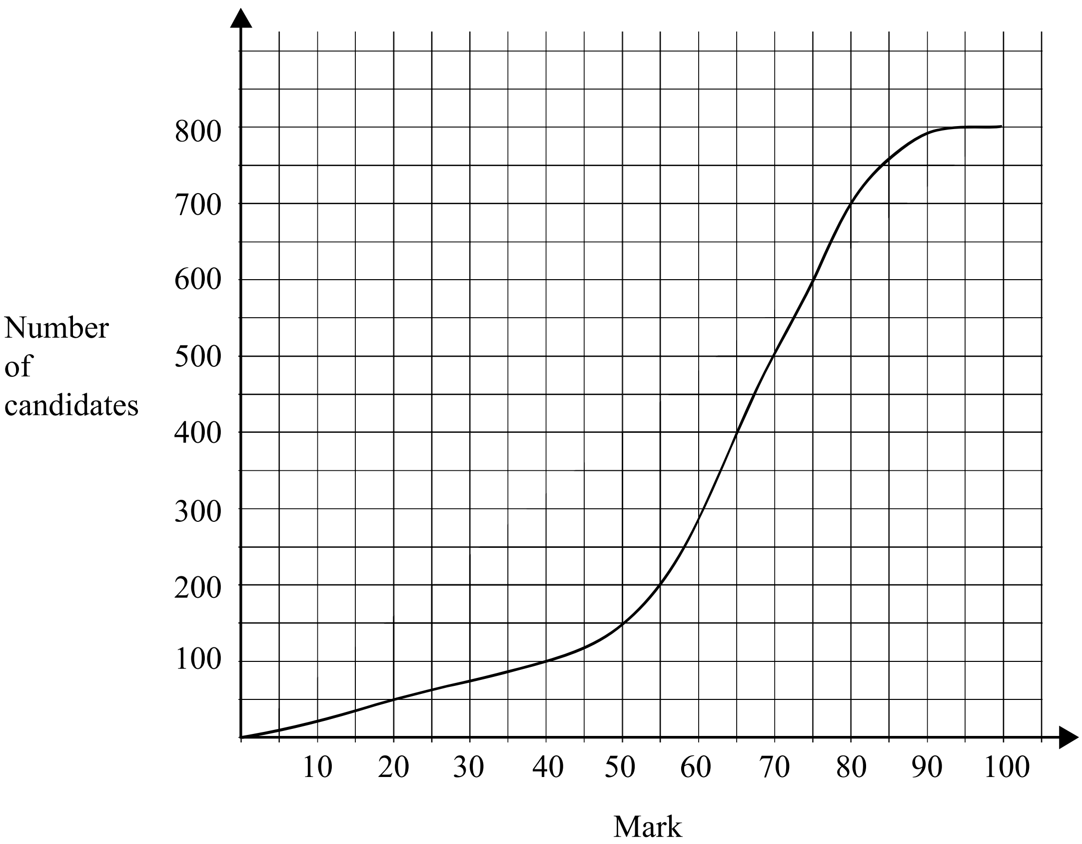
If $\mathrm{P}(A) = \frac{1}{6}$, $\mathrm{P}(B) = \frac{1}{3}$, and $\mathrm{P}(A \cup B) = \frac{5}{12}$, what is $\mathrm{P}(A' / B')$?

27. *[Maximum mark: 5]*

A room has nine desks arranged in three rows of three desks. Three students sit in the room. If the students randomly choose a desk find the probability that two out of the front three desks are chosen.

28. *[Maximum mark: 6]*

A test marked out of 100 is written by 800 students. The cumulative frequency graph for the marks is given below.



(a) Write down the number of students who scored 40 marks or less on the test. *[2 marks]*

(b) The middle 50 % of test results lie between marks a and b, where $a < b$. Find a and b. *[4 marks]*

29. *[Maximum mark: 6]*

A discrete random variable X has its probability distribution given by

$$\mathrm{P}(X = x) = k(x+1), \text{where } x \text{ is } 0, 1, 2, 3, 4.$$

(a) Show that $k = \frac{1}{15}$. *[3 marks]*

(b) Find $\mathrm{E}(X)$. *[3 marks]*

30. *[Maximum mark: 6]*

The function f' is given by $f'(x) = 2\sin\left(5x - \frac{\pi}{2}\right)$.

(a) Write down $f''(x)$. *[2 marks]*

(b) Given that $f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$. *[4 marks]*

31. *[Maximum mark: 6]*

Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point $(1, -2)$.

32. *[Maximum mark: 6]*

Solve the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 1$, given that $y = 0$ when $x = 2$. Give your answer in the form $y = f(x)$.

33. *[Maximum mark: 7]*

(a) Sketch the curves $y = x^2$ and $y = |x|$. *[3 marks]*

(b) Find the sum of the areas of the regions enclosed by the curves $y = x^2$ and $y = |x|$. *[4 marks]*

34. *[Maximum mark: 9]*

The acceleration of a body is given in terms of the displacement s metres as $a = \frac{2s}{s^2 + 1}$.

(a) Give a formula for the velocity as a function of the displacement given that when $s = 1$ metre, $v = 2\ \text{ms}^{-1}$. *[7 marks]*

(b) Hence find the velocity when the body has travelled 5 metres. *[2 marks]*

35. *[Maximum mark: 7]*

A curve C is defined implicitly by $x\mathrm{e}^{y} = x^2 + y^2$. Find the equation of the tangent to C at the point $(1, 0)$.

36. *[Maximum mark: 9]*

The function f is defined by $f(x) = \left(\ln(x-2)\right)^2$. Find the coordinates of the point of inflexion of f.

37. *[Maximum mark: 5]*

Find $\int_1^{\mathrm{e}} \frac{(\ln x)^3}{x}\,\mathrm{d}x$.

38. *[Maximum mark: 7]*

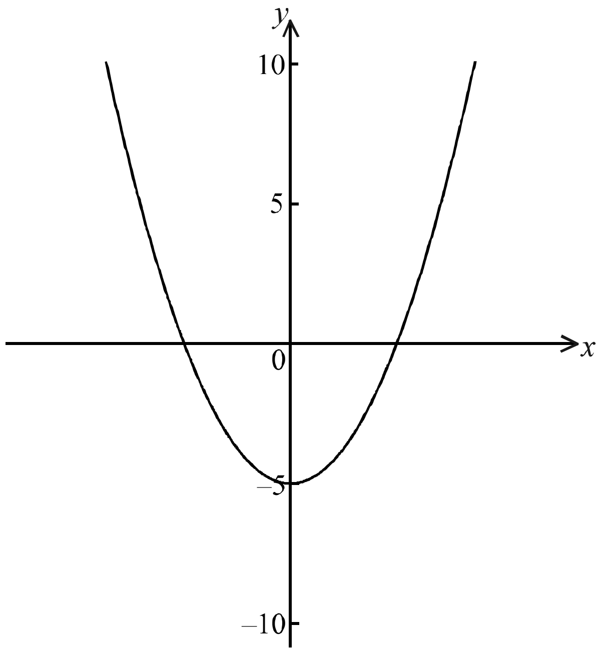
Find the value of the integral $\int_0^4 \left| x^2 - 4 \right| \mathrm{d}x$.

39. *[Maximum mark: 11]*

Find $\int_1^{\sqrt{3}} \sqrt{4 - x^2}\,\mathrm{d}x$ using the substitution $x = 2\sin\theta$.

40. *[Maximum mark: 7]*

The curve $y = x^2 - 5$ is shown below.



A point P on the curve has x-coordinate equal to a.

(a) Show that the distance OP is $\sqrt{a^4 - 9a^2 + 25}$. *[2 marks]*

(b) Find the values of a for which the curve is closest to the origin. *[5 marks]*

41. *[Maximum mark: 7]*

Find $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}} \mathrm{d}x$.

42. *[Maximum mark: 6]*

Use the substitution $u = x + 2$ to find $\int \frac{x^3}{(x+2)^2} \mathrm{d}x$.

Section B questions

43. *[Maximum mark: 22]*

(a) Show that $p = 2$ is a solution to the equation $p^3 + p^2 - 5p - 2 = 0$. *[2 marks]*

(b) Find the values of a and b such that $p^3 + p^2 - 5p - 2 = (p-2)(p^2 + ap + b)$. *[4 marks]*

(c) Hence find the other two roots to the equation $p^3 + p^2 - 5p - 2 = 0$. *[3 marks]*

(d) An arithmetic sequence has p as its common difference. Also, a geometric sequence has p as its common ratio. Both sequences have 1 as their first term.

(i) Write down, in terms of p, the first four terms of each sequence.

(ii) If the sum of the third and fourth terms of the arithmetic sequence is equal to the sum of the third and fourth terms of the geometric sequence, find the three possible values of p.

(iii) For which value of p found in (d)(ii) does the sum to infinity of the terms of the geometric sequence exist?

(iv) For the same value p, find the sum of the first 20 terms of the arithmetic sequence writing your answer in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Z}$. *[13 marks]*

44. *[Total mark: 25]*

Part A *[Maximum mark: 9]*

Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$. *[9 marks]*

Part B *[Maximum mark: 16]*

Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$.

(a) Find an expression for z, the common ratio of this series. *[2 marks]*

(b) Show that $|z| < 1$. *[2 marks]*

(c) Write down an expression for the sum to infinity of this series. *[2 marks]*

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$

[10 marks]

45. *[Maximum mark: 31]*

The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β.

(a) Find α and β in the form $re^{i\theta}$. *[6 marks]*

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram. *[2 marks]*

(c) Use the principle of mathematical induction to prove De Moivre's theorem which states that $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ for $n \in \mathbb{Z}^+$. *[8 marks]*

(d) Using De Moivre's theorem find $\frac{\alpha^3}{\beta^2}$ in the form $a + ib$. *[4 marks]*

(e) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$. *[3 marks]*

(f) Find the exact value of $\alpha\beta^* + \beta\alpha^*$ where α^* is the conjugate of α and β^* is the conjugate of β. *[5 marks]*

(g) Find the set of values of n for which α^n is real. *[3 marks]*

46. *[Maximum mark: 13]*

The lengths of the sides of a triangle ABC are $x-2$, x and $x+2$. The largest angle is 120°.

(a) Find the value of x. *[6 marks]*

(b) Show that the area of the triangle is $\frac{15\sqrt{3}}{4}$. *[3 marks]*

(c) Find $\sin A+\sin B+\sin C$ giving your answer in the form $\frac{p\sqrt{q}}{r}$ where $p, q, r \in \mathbb{Z}$. *[4 marks]*

47. *[Maximum mark: 13]*

(a) Show that the following system of equations will have a unique solution when $a \neq -1$.

$$x+3y-z=0$$
$$3x+5y-z=0$$
$$x-5y+(2-a)z=9-a^2$$

[5 marks]

(b) State the solution in terms of a. *[6 marks]*

(c) Hence, solve

$$x+3y-z=0$$
$$3x+5y-z=0$$
$$x-5y+z=8$$

[2 marks]

48. *[Maximum mark: 25]*

Consider the points $A(1, 2, 1)$, $B(0, -1, 2)$, $C(1, 0, 2)$ and $D(2, -1, -6)$.

(a) Find the vectors $\vec{AB}$ and $\vec{BC}$. *[2 marks]*

(b) Calculate $\vec{AB} \times \vec{BC}$. *[2 marks]*

(c) Hence, or otherwise find the area of triangle ABC. *[3 marks]*

(d) Find the Cartesian equation of the plane *P* containing the points A, B and C. *[3 marks]*

(e) Find a set of parametric equations for the line *L* through the point D and perpendicular to the plane *P*. *[3 marks]*

(f) Find the point of intersection E, of the line *L* and the plane *P*. *[4 marks]*

(g) Find the distance from the point D to the plane *P*. *[2 marks]*

(h) Find a unit vector which is perpendicular to the plane *P*. *[2 marks]*

(i) The point F is a reflection of D in the plane *P*. Find the coordinates of F. *[4 marks]*

49. *[Maximum mark: 29]*

(a) Show that lines $\frac{x-2}{1} = \frac{y-2}{3} = \frac{z-3}{1}$ and $\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-4}{2}$ intersect and find the coordinates of P, the point of intersection. *[8 marks]*

(b) Find the Cartesian equation of the plane Π that contains the two lines. *[6 marks]*

(c) The point $Q(3, 4, 3)$ lies on Π. The line *L* passes through the midpoint of [PQ]. Point S is on *L* such that $\left|\overrightarrow{PS}\right| = \left|\overrightarrow{QS}\right| = 3$, and the triangle PQS is normal to the plane Π. Given that there are two possible positions for S, find their coordinates. *[15 marks]*

50. *[Maximum mark: 20]*

The probability density function of the random variable X is given by

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{4-x^2}}, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant k. *[5 marks]*

(b) Show that $\mathrm{E}(X) = \dfrac{6(2-\sqrt{3})}{\pi}$. *[7 marks]*

(c) Determine whether the median of X is less than $\dfrac{1}{2}$ or greater than $\dfrac{1}{2}$. *[8 marks]*

51. *[Maximum mark: 13]*

Bag A contains 2 red and 3 green balls.

(a) Two balls are chosen at random from the bag without replacement. Find the probability that 2 red balls are chosen. *[2 marks]*

Bag B contains 4 red and n green balls.

(b) Two balls are chosen without replacement from this bag. If the probability that two red balls are chosen is $\dfrac{2}{15}$, show that $n = 6$. *[4 marks]*

A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.

(c) Calculate the probability that two red balls are chosen. *[3 marks]*

(d) Given that two red balls are chosen, find the probability that a 1 or a 6 was obtained on the die. *[4 marks]*

52. *[Maximum mark: 14]*

It is given that

$$f(x)=\frac{18(x-1)}{x^2},\ f'(x)=\frac{18(2-x)}{x^3},\text{ and } f''(x)=\frac{36(x-3)}{x^4},\ x\in\mathbb{R},\ x\neq 0.$$

(a) Find

(i) the zero(s) of $f(x)$;

(ii) the equations of the asymptotes;

(iii) the coordinates of the local maximum and justify it is a maximum;

(iv) the interval(s) where $f(x)$ is concave up. *[9 marks]*

(b) Hence sketch the graph of $y = f(x)$. *[5 marks]*

53. *[Maximum mark: 18]*

The function f is defined on the domain $x \geq 1$ by $f(x)=\dfrac{\ln x}{x}$.

(a) (i) Show, by considering the first and second derivatives of *f*, that there is one maximum point on the graph of *f*.

(ii) State the **exact** coordinates of this point.

(iii) The graph of f has a point of inflexion at P. Find the x-coordinate of P. *[12 marks]*

Let R be the region enclosed by the graph of *f*, the x-axis and the line $x = 5$.

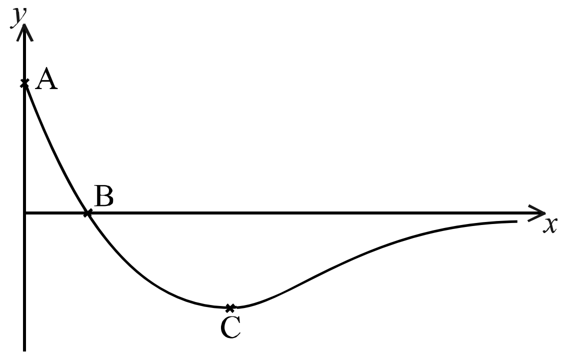
(b) Find the **exact** value of the area of R. *[6 marks]*

54. *[Maximum mark: 16]*

(a) Find the root of the equation $e^{2-2x} = 2e^{-x}$ giving the answer as a logarithm. *[4 marks]*

(b) The curve $y = e^{2-2x} - 2e^{-x}$ has a minimum point. Find the coordinates of this minimum. *[7 marks]*

(c) The curve $y = e^{2-2x} - 2e^{-x}$ is shown below.



Write down the coordinates of the points A, B and C. *[3 marks]*

(d) Hence state the set of values of k for which the equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive roots. *[2 marks]*

55. *[Maximum mark: 21]*

The function f is defined on the domain $x \geq 0$ by $f(x) = \frac{x^2}{e^x}$.

(a) Find the maximum value of $f(x)$, and justify that it is a maximum. *[10 marks]*

(b) Find the x coordinates of the points of inflexion on the graph of f. *[3 marks]*

(c) Evaluate $\int_0^1 f(x)\,dx$. *[8 marks]*

Paper 1 markscheme

Section A

1. **EITHER**

$4\ln 2 - 3\ln 2^2 = -\ln k$ ***M1***

$4\ln 2 - 6\ln 2 = -\ln k$ ***(M1)***

$-2\ln 2 = -\ln k$ ***(A1)***

$-\ln 2^2 = -\ln k$ ***M1***

$k = 4$ ***A1***

OR

$\ln 2^4 - \ln 4^3 = -\ln k$ ***M1***

$\ln \frac{2^4}{4^3} = \ln k^{-1}$ ***M1A1***

$\frac{2^4}{4^3} = \frac{1}{k}$ ***A1***

$\Rightarrow k = \frac{4^3}{2^4} = \frac{64}{16} = 4$ ***A1***

[5 marks]

2. $\log_3(x+17) - 2 = \log_3 2x$

$\log_3(x+17) - \log_3 2x = 2$

$\log_3\left(\frac{x+17}{2x}\right) = 2$ ***M1A1***

$\frac{x+17}{2x} = 9$ ***M1A1***

$x + 17 = 18x$

$17 = 17x$

$x = 1$ ***A1***

[5 marks]

3. $2^{2x+2} - 10 \times 2^x + 4 = 0$

$y = 2^x$

$4y^2 - 10y + 4 = 0$ ***M1A1***

$2y^2 - 5y + 2 = 0$

By factorisation or using the quadratic formula ***(M1)***

$y = \frac{1}{2} \qquad y = 2$ ***A1***

$2^x = \frac{1}{2} \qquad 2^x = 2$

$x = -1 \qquad x = 1$ ***A1A1***

[6 marks]

4. $a^2 + 2\mathrm{i}ab - b^2 = 3 + 4\mathrm{i}$

Equate real and imaginary parts *(M1)*

$a^2 - b^2 = 3$, $2ab = 4$ ***A1***

Since $b = \dfrac{2}{a}$

$\Rightarrow a^2 - \dfrac{4}{a^2} = 3$ *(M1)*

$\Rightarrow a^4 - 3a^2 - 4 = 0$ ***A1***

Using factorisation or the quadratic formula *(M1)*

$\Rightarrow a = \pm 2$

$\Rightarrow b = \pm 1$

$\Rightarrow \sqrt{3 + 4\mathrm{i}} = 2 + \mathrm{i}, -2 - \mathrm{i}$ ***A1A1***

[7 marks]

5. $2 + \mathrm{i}$ is a root $\Rightarrow$ $2 - \mathrm{i}$ is a root ***R1***

$[x - (2 + \mathrm{i})]\,[x - (2 - \mathrm{i})]$ are factors ***M1***

$= x^2 - (2 - \mathrm{i})x - (2 + \mathrm{i})x + (2 + \mathrm{i})(2 - \mathrm{i})$

$= x^2 - 2x + \mathrm{i}x - 2x - \mathrm{i}x + (4 + 1)$ *(A1)*

$= x^2 - 4x + 5$ ***A1***

Hence $x - 2$ is a factor $\Rightarrow 2$ is a root ***R1***

[5 marks]

6. $5zz^* + 10 = (6 - 18\mathrm{i})z^*$ ***M1***

Let $z = a + \mathrm{i}b$

$5 \times 10 + 10 = (6 - 18\mathrm{i})(a - b\mathrm{i})$ $(= 6a - 6b\mathrm{i} - 18a\mathrm{i} - 18b)$ ***M1A1***

Equate real and imaginary parts *(M1)*

$\Rightarrow$ $6a - 18b = 60$ and $6b + 18a = 0$

$\Rightarrow$ $a = 1$ and $b = -3$ ***A1A1***

$z = 1 - 3\mathrm{i}$ ***A1***

[7 marks]

7. $8\mathrm{i} = 8\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2n\pi\right)}$ ***(M1)***

For $n = 0$

$(8\mathrm{i})^{\frac{1}{3}} = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***(M1)***

$= 2\cos\dfrac{\pi}{6} + 2\mathrm{i}\sin\dfrac{\pi}{6}$ ***A1***

$= \sqrt{3} + \mathrm{i}$ ***A1***

For $n = 1$

$(8\mathrm{i})^{\frac{1}{3}} = 2\cos\dfrac{5\pi}{6} + 2\mathrm{i}\sin\dfrac{5\pi}{6}$ ***M1***

$= -\sqrt{3} + \mathrm{i}$ ***A1***

For $n = 2$

$(8\mathrm{i})^{\frac{1}{3}} = 2\cos\dfrac{3\pi}{2} + 2\mathrm{i}\sin\dfrac{3\pi}{2}$ ***M1***

$= -2\mathrm{i}$ ***A1***

[8 marks]

8. $\mathrm{i}z_1 + 2z_2 = 3 \Rightarrow z_2 = -\frac{1}{2}\mathrm{i}z_1 + \frac{3}{2}$

$z_1 + (1-\mathrm{i})z_2 = 4$

$\Rightarrow z_1 + (1-\mathrm{i})\left(-\frac{1}{2}\mathrm{i}z_1 + \frac{3}{2}\right) = 4$ ***M1A1***

$\Rightarrow z_1 - \frac{1}{2}\mathrm{i}z_1 + \frac{3}{2} + \frac{1}{2}\mathrm{i}^2 z_1 - \frac{3}{2}\mathrm{i} = 4$

$\Rightarrow \frac{1}{2}z_1 - \frac{1}{2}\mathrm{i}z_1 = \frac{5}{2} + \frac{3}{2}\mathrm{i}$

$\Rightarrow z_1 - \mathrm{i}z_1 = 5 + 3\mathrm{i}$ ***A1***

EITHER

Let $z_1 = x + \mathrm{i}y$ ***(M1)***

$\Rightarrow x + \mathrm{i}y - \mathrm{i}x - \mathrm{i}^2 y = 5 + 3\mathrm{i}$

Equate real and imaginary parts ***(M1)***

$\Rightarrow x + y = 5$

$-x + y = 3$

$2y = 8$

$y = 4 \Rightarrow x = 1$ *i.e.* $z_1 = 1 + 4\mathrm{i}$ ***A1A1***

$z_2 = -\frac{1}{2}\mathrm{i}(1+4\mathrm{i}) + \frac{3}{2}$ ***M1***

$z_2 = -\frac{1}{2}\mathrm{i} - 2\mathrm{i}^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}\mathrm{i}$ ***A1***

OR

$z_1 = \frac{5+3\mathrm{i}}{1-\mathrm{i}}$ ***M1***

$z_1 = \frac{(5+3\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})} \quad \left(= \frac{5+8\mathrm{i}-3}{2}\right)$ ***M1A1***

$z_1 = 1 + 4\mathrm{i}$ ***A1***

$z_2 = -\frac{1}{2}\mathrm{i}(1+4\mathrm{i}) + \frac{3}{2}$ ***M1***

$z_2 = -\frac{1}{2}\mathrm{i} - 2\mathrm{i}^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}\mathrm{i}$ ***A1***

[9 marks]

9. **METHOD 1**

$20+10b\text{i}=(1-b\text{i})(-7+9\text{i})$ ***(M1)***

$20+10b\text{i}=(-7+9b)+(9+7b)\text{i}$ ***A1A1***

Equate real and imaginary parts ***(M1)***

EITHER

$-7+9b=20$

$b=3$ ***(M1)A1***

OR

$10b=9+7b$

$3b=9$

$b=3$ ***(M1)A1***

METHOD 2

$\frac{(2+b\text{i})(1+b\text{i})}{(1-b\text{i})(1+b\text{i})}=\frac{-7+9\text{i}}{10}$ ***(M1)***

$\frac{2-b^2+3b\text{i}}{1+b^2}=\frac{-7+9\text{i}}{10}$ ***A1***

Equate real and imaginary parts ***(M1)***

$\frac{2-b^2}{1+b^2}=-\frac{7}{10}$ Equation A

$\frac{3b}{1+b^2}=\frac{9}{10}$ Equation B

From equation A

$20-10b^2=-7-7b^2$

$3b^2=27$

$b=\pm 3$ ***A1***

From equation B

$30b=9+9b^2$

$3b^2-10b+3=0$

By factorisation or using the quadratic formula

$b=\frac{1}{3}$ or 3 ***A1***

Since 3 is the common solution to both equations $b=3$ ***R1***

[6 marks]

10. METHOD 1

since $b > 0$ *(M1)*

$\Rightarrow \arg(b+\mathrm{i}) = 30^\circ$ *A1*

$\frac{1}{b} = \tan 30^\circ$ *M1A1*

$b = \sqrt{3}$ *A2* *N2*

[6 marks]

METHOD 2

$\arg(b+\mathrm{i})^2 = 60^\circ \Rightarrow \arg(b^2 - 1 + 2b\mathrm{i}) = 60^\circ$ *M1*

$\frac{2b}{(b^2-1)} = \tan 60^\circ = \sqrt{3}$ *M1A1*

$\sqrt{3}b^2 - 2b - \sqrt{3} = 0$ *A1*

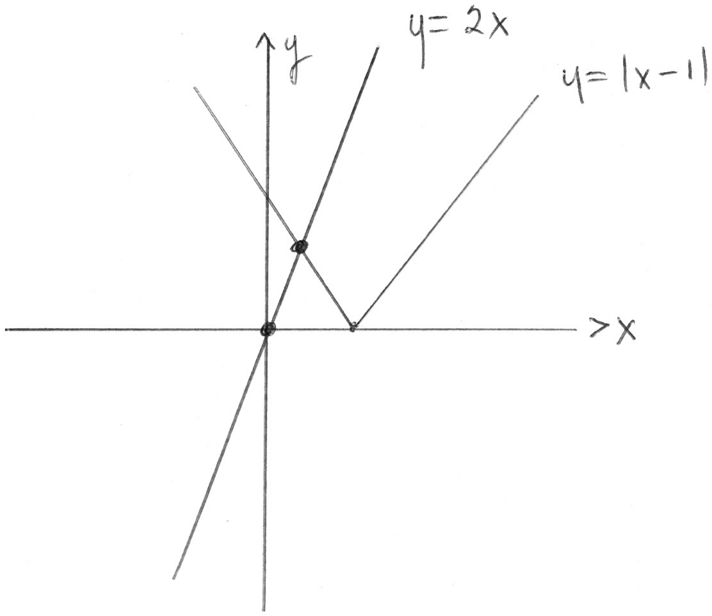
$\left(\sqrt{3}b+1\right)\left(b-\sqrt{3}\right) = 0$

since $b > 0$ *(M1)*

$b = \sqrt{3}$ *A1* *N2*

[6 marks]

11.



A1A1

Note: Award ***A1*** for each graph.

$2x = 1 - x \quad \Rightarrow \quad x = \frac{1}{3}$ *M1A1*

$\therefore \quad x < \frac{1}{3}$ *A1*

[5 marks]

12. Attempting to find $f(2) = 8 + 12 + 2a + b$ ***(M1)***

$= 2a + b + 20$ ***A1***

Attempting to find $f(-1) = -1 + 3 - a + b$ ***(M1)***

$= 2 - a + b$ ***A1***

Equating $2a + 20 = 2 - a$ ***A1***

$a = -6$ ***A1*** ***N2***

[6 marks]

13. (a) $f : x \mapsto \mathrm{e}^x \Rightarrow f^{-1} : x \mapsto \ln x$

$\Rightarrow f^{-1}(3) = \ln 3$ ***A1***

$g : x \mapsto x + 2 \Rightarrow g^{-1} : x \mapsto x - 2$

$\Rightarrow g^{-1}(3) = 1$ ***A1***

$f^{-1}(3) \times g^{-1}(3) = \ln 3$ ***A1*** ***N1***

[3 marks]

(b) **EITHER**

$f \circ g(x) = f(x + 2) = \mathrm{e}^{x+2}$ ***A1***

$\mathrm{e}^{x+2} = 3 \Rightarrow x + 2 = \ln 3$ ***M1***

$x = \ln 3 - 2$ ***A1*** ***N0***

[3 marks]

OR

$f \circ g(x) = \mathrm{e}^{x+2}$

$f \circ g^{-1}(x) = \ln(x) - 2$ ***A1***

$f \circ g^{-1}(3) = \ln(3) - 2$ ***M1***

$x = \ln 3 - 2$ ***A1*** ***N0***

[3 marks]

Total [6 marks]

14. $2\sin x \cos x - \sqrt{2}\cos x = 0$ ***(M1)***

$\cos x\left(2\sin x - \sqrt{2}\right) = 0$ ***(A1)***

$\cos x = 0 \qquad \sin x = \frac{\sqrt{2}}{2}$ ***A1***

$x = \frac{\pi}{2} \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}$ ***A1A1A1***

[6 marks]

15. (a) $\sin B = \frac{5}{13}$ *A1*

[1 mark]

(b) $\cos B = -\frac{12}{13}$ *A1*

[1 mark]

(c) $\sin 2B = 2\sin B \cos B$ *(M1)*

$= 2 \times \frac{5}{13} \times -\frac{12}{13}$

$= -\frac{120}{169}$ *A1*

[2 marks]

(d) $\cos 2B = 2\cos^2 B - 1$ *(M1)*

$= 2\left(\frac{144}{169}\right) - 1$

$= \frac{119}{169}$ *A1*

[2 marks]

Total [6 marks]

16. Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ *(M1)*

$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{3}{4}$

$3\tan^2\theta + 8\tan\theta - 3 = 0$ *A1*

Using factorisation or the quadratic formula *(M1)*

$\tan\theta = \frac{1}{3}$ or -3 *A1A1*

[5 marks]

17. (a) $4(1-2s^2) - 3s\frac{1}{s^3} + 6 = 0$ *M1A1*

$4s^2 - 8s^4 + 6s^2 - 3 = 0$ *A1*

$8s^4 - 10s^2 + 3 = 0$ *AG*

[3 marks]

(b) Attempt to factorise or use the quadratic formula *(M1)*

$\sin^2 x = \frac{1}{2}$ or $\sin^2 x = \frac{3}{4}$ *(A1)*

$\sin x = \frac{\sqrt{2}}{2} \Rightarrow x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$ *A1A1*

$\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}$ *A1A1*

Note: Penalise ***A1*** if extraneous solutions given.

[6 marks]

Total [9 marks]

18. (a) $\sin x\cos\alpha - \cos x\sin\alpha = k\sin x\cos\alpha + k\cos x\sin\alpha$ *(M1)*

$\Rightarrow \tan x\cos\alpha - \sin\alpha = k\tan x\cos\alpha + k\sin\alpha$ ***M1***

$\Rightarrow \tan x = \dfrac{-(k+1)\sin\alpha}{(k-1)\cos\alpha}\left(= \dfrac{-(k+1)}{(k-1)}\tan\alpha\right)$ ***A1***

[3 marks]

(b) $\tan x = \dfrac{-\frac{3}{2}\sin 210^\circ}{-\frac{1}{2}\cos 210^\circ}$ *(M1)*

Now $\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$ and $\cos 210^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$ ***A1A1***

$\tan x = \dfrac{3\times -\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \dfrac{3}{2}\times\dfrac{2}{\sqrt{3}} = \dfrac{3}{\sqrt{3}} = \sqrt{3}$ ***A1***

$\Rightarrow x = 60^\circ,\ 240^\circ$ ***A1A1***

[6 marks]

Total [9 marks]

19. $2\tan^2\theta - 5\sec\theta - 10 = 0$

Using $1+\tan^2\theta = \sec^2\theta$, $\Rightarrow 2(\sec^2\theta - 1) - 5\sec\theta - 10 = 0$ *(M1)*

$2\sec^2\theta - 5\sec\theta - 12 = 0$ ***A1***

Solving the equation *e.g.* $(2\sec\theta + 3)(\sec\theta - 4) = 0$ *(M1)*

$\sec\theta = -\frac{3}{2}$ or $\sec\theta = 4$ ***A1***

θ in second quadrant $\Rightarrow \sec\theta$ is negative *(R1)*

$\Rightarrow \sec\theta = -\frac{3}{2}$ ***A1*** ***N3***

[6 marks]

20. $\det \boldsymbol{A} = 3\begin{vmatrix} 5 & 8 \\ 4 & 6 \end{vmatrix} - 1\begin{vmatrix} 9 & 8 \\ 7 & 6 \end{vmatrix} + 2\begin{vmatrix} 9 & 5 \\ 7 & 4 \end{vmatrix}$ ***M1***

$= 3(30-32) - 1(54-56) + 2(36-35)$ *(A1)(A1)(A1)*

$= 3(-2) - 1(-2) + 2(1)$

$= -6 + 2 + 2 \quad (= -2)$ ***A1***

[5 marks]

21. $\boldsymbol{A}^2 = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} 1+2k & 0 \\ 0 & 2k+1 \end{pmatrix}$ ***A2***

Note: Award ***A2*** for 4 correct, ***A1*** for 2 or 3 correct.

$1+2k=0$ ***M1***

$k=-\frac{1}{2}$ ***A1***

[5 marks]

22. $\boldsymbol{M}^2 = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix}$ ***M1A1***

$\Rightarrow \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix} - \begin{pmatrix} 12 & -6 \\ -18 & 24 \end{pmatrix} + k\boldsymbol{I} = 0$ ***(M1)***

$\Rightarrow \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} + k\boldsymbol{I} = 0$ ***(A1)***

$\Rightarrow k = 5$ ***A1***

[5 marks]

23. For multiplying $(\boldsymbol{I}-\boldsymbol{X})(\boldsymbol{I}+\boldsymbol{X}+\boldsymbol{X}^2)$ ***M1***

$= \boldsymbol{I}^2 + \boldsymbol{IX} + \boldsymbol{IX}^2 - \boldsymbol{XI} - \boldsymbol{X}^2 - \boldsymbol{X}^3 = \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2 - \boldsymbol{X} - \boldsymbol{X}^2 - \boldsymbol{X}^3$ ***(A1)(A1)***

$= \boldsymbol{I} - \boldsymbol{X}^3$ ***A1***

$= \boldsymbol{I}$ ***A1***

$\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{B}$ ***(R1)***

$(\boldsymbol{I}-\boldsymbol{X})(\boldsymbol{I}+\boldsymbol{X}+\boldsymbol{X}^2) = \boldsymbol{I} \Rightarrow (\boldsymbol{I}-\boldsymbol{X})^{-1} = \boldsymbol{I}+\boldsymbol{X}+\boldsymbol{X}^2$ ***AG*** ***N0***

[6 marks]

24. **EITHER**

Let s be the distance from the origin to a point on the line, then

$s^2 = (1-\lambda)^2 + (2-3\lambda)^2 + 4$ ***(M1)***

$= 10\lambda^2 - 14\lambda + 9$ ***A1***

$\frac{d(s^2)}{d\lambda} = 20\lambda - 14$ ***A1***

For minimum $\frac{d(s^2)}{d\lambda} = 0, \Rightarrow \lambda = \frac{7}{10}$ ***A1***

OR

The position vector for the point nearest to the origin is perpendicular to the direction of the line. At that point:

$\begin{pmatrix} 1-\lambda \\ 2-3\lambda \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = 0$ ***(M1)A1***

Therefore, $10\lambda - 7 = 0$ ***A1***

Therefore, $\lambda = \frac{7}{10}$ ***A1***

THEN

$x = \frac{3}{10},\ y = -\frac{1}{10}$ ***(A1)(A1)***

The point is $\left(\frac{3}{10}, \frac{-1}{10}, 2\right)$. ***N3***

[6 marks]

25. $X \sim \text{Po}(\mu)$

$P(X=10) = 2P(X=9)$ ***(M1)***

$\frac{e^{-\mu}\mu^{10}}{10!} = \frac{2e^{-\mu}\mu^{9}}{9!}$ ***A1A1***

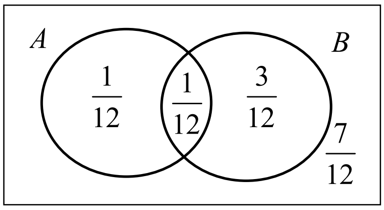
$\mu = \frac{10! \times 2}{9!} = 10 \times 2 = 20$ ***A1***

$E(X) = 20$ ***A1***

[5 marks]

26. $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ ***M1***

$= \frac{2}{12} + \frac{4}{12} - \frac{5}{12} = \frac{1}{12}$ ***A1***



M1A1

$P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{\frac{7}{12}}{\frac{8}{12}} = \frac{7}{8}$ ***M1A1***

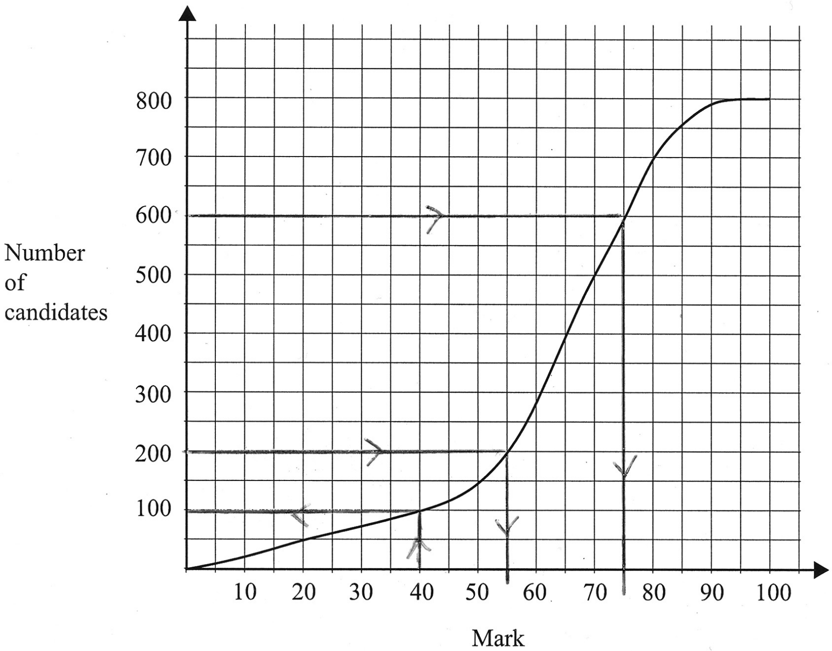
[6 marks]

27. Probability $= \frac{{}^{3}C_{2} \times {}^{6}C_{1}}{{}^{9}C_{3}}$ ***M1A1A1A1***

$= \frac{3 \times 6 \times 3! \times 6!}{9!} = \frac{3 \times 6 \times 6}{9 \times 8 \times 7} = \frac{3}{14}$ ***A1***

[5 marks]

28. (a)



Lines on graph ***(M1)***

100 students score 40 marks or fewer. ***A1*** ***N2***

[2 marks]

(b) Identifying 200 **and** 600 ***A1***

Lines on graph. ***(M1)***

$a = 55, \ b = 75$ ***A1A1*** ***N1N1***

[4 marks]

Total [6 marks]

29. (a) Using $\sum \mathrm{P}(X = x) = 1$ *(M1)*

$\therefore\ k \times 1 + k \times 2 + k \times 3 + k \times 4 + k \times 5 = 15k = 1$ *M1A1*

$k = \frac{1}{15}$ *AG* *N0*

[3 marks]

(b) Using $\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$ *(M1)*

$= 0 \times \frac{1}{15} + 1 \times \frac{2}{15} + 2 \times \frac{3}{15} + 3 \times \frac{4}{15} + 4 \times \frac{5}{15}$ *A1*

$= \frac{8}{3}\left(2\frac{2}{3},\ 2.67\right)$ *A1* *N2*

[3 marks]

Total [6 marks]

30. (a) Using the chain rule $f''(x) = \left(2\cos\left(5x - \frac{\pi}{2}\right)\right)5$ *(M1)*

$= 10\cos\left(5x - \frac{\pi}{2}\right)$ *A1* *N2*

[2 marks]

(b) $f(x) = \int f'(x)\,\mathrm{d}x$

$= -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + c$ *A1*

Substituting to find c, $f\left(\frac{\pi}{2}\right) = -\frac{2}{5}\cos\left(5\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right) + c = 1$ *M1*

$c = 1 + \frac{2}{5}\cos 2\pi = 1 + \frac{2}{5} = \frac{7}{5}$ *(A1)*

$f(x) = -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + \frac{7}{5}$ *A1* *N2*

[4 marks]

Total [6 marks]

31. Attempting to differentiate implicitly ***(M1)***

$3x^2y + 2xy^2 = 2 \Rightarrow 6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2 + 4xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1***

Substituting $x = 1$ and $y = -2$ ***(M1)***

$-12 + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 8 - 8\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1***

$\Rightarrow -5\frac{\mathrm{d}y}{\mathrm{d}x} = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{5}$ ***A1***

Gradient of normal is $\frac{5}{4}$ ***A1*** ***N3***

[6 marks]

32. $x\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 1, \Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 1$

Separating variables ***(M1)***

$\frac{\mathrm{d}y}{y^2+1} = \frac{\mathrm{d}x}{x}$ ***A1***

$\arctan y = \ln x + c$ ***A1A1***

$y = 0,\ x = 2 \Rightarrow \arctan 0 = \ln 2 + c$

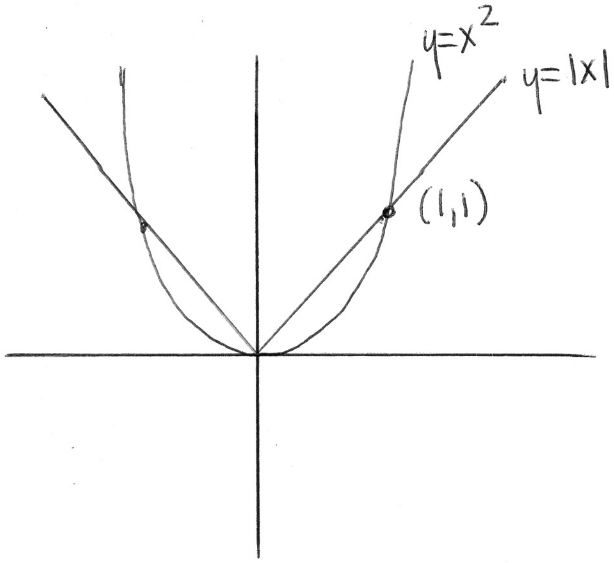
$-\ln 2 = c$ ***(A1)***

$\arctan y = \ln x - \ln 2 = \ln\frac{x}{2}$

$y = \tan\left(\ln\frac{x}{2}\right)$ ***A1*** ***N3***

[6 marks]

33. (a)



A1A1A1

Note:	Award ***A1*** for correct shape, ***A1*** for points of intersection and ***A1*** for symmetry.

[3 marks]

(b) $A = 2\int_0^1 (x - x^2)\,\mathrm{d}x$ ***M1***

$= 2\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1$ ***A1***

$= 2\left(\dfrac{1}{2} - \dfrac{1}{3}\right)$ ***(A1)***

$= \dfrac{1}{3}$ square units ***A1***

[4 marks]

Total [7 marks]

34. (a) $a = \frac{2s}{s^2+1}$

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$ ***M1***

$v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{2s}{s^2+1}$

$\int v\,\mathrm{d}v = \int \frac{2s}{s^2+1}\mathrm{d}s$ ***M1***

$\Rightarrow \frac{v^2}{2} = \ln\left|s^2+1\right| + k$ ***A1A1***

Note: Do not penalize if k is missing.

When $s = 1$, $v = 2$

$\Rightarrow 2 = \ln 2 + k$ ***M1***

$\Rightarrow k = 2 - \ln 2$ ***A1***

$\Rightarrow \frac{v^2}{2} = \ln\left|s^2+1\right| + 2 - \ln 2 \quad \left(= \ln\left|\frac{s^2+1}{2}\right| + 2\right)$ ***A1***

[7 marks]

(b) **EITHER**

$\frac{v^2}{2} = \ln\left|\frac{26}{2}\right| + 2$ ***M1***

$\Rightarrow v^2 = 2\ln\left|13\right| + 4$

$\Rightarrow v = \sqrt{2\ln\left|13\right| + 4}$ ***A1***

OR

$\frac{v^2}{2} = \ln\left|26\right| + 2 - \ln 2$ ***M1***

$v^2 = 2\ln\left|26\right| + 4 - 2\ln 2$

$v = \sqrt{2\ln\left|26\right| + 4 - 2\ln 2}$ ***A1***

[2 marks]

Total [9 marks]

35. $x\mathrm{e}^y = x^2 + y^2$

$\mathrm{e}^y + x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1A1A1A1***

$(1, 0)$ fits $\Rightarrow 1 + \frac{\mathrm{d}y}{\mathrm{d}x} = 2 + 0$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1$ ***A1***

Equation of tangent is $y = x + c$

$(1, 0)$ fits $\Rightarrow c = -1$

$\Rightarrow y = x - 1$ ***A1***

[7 marks]

36. $f'(x) = \dfrac{2\left(\ln(x-2)\right)}{x-2}$ ***M1A1***

$f''(x) = \dfrac{(x-2)\left(\dfrac{1}{x-2}\right) - 2\ln(x-2)\times 1}{(x-2)^2}$ ***M1A1***

$= \dfrac{2 - 2\ln(x-2)}{(x-2)^2}$ ***A1***

$f''(x) = 0$ for point of inflexion ***(M1)***

$\Rightarrow 2 - 2\ln(x-2) = 0$

$\ln(x-2) = 1$ ***A1***

$x - 2 = \mathrm{e}$

$x = \mathrm{e} + 2$ ***A1***

$\Rightarrow f(x) = \left(\ln(\mathrm{e}+2-2)\right)^2 = (\ln \mathrm{e})^2 = 1$ ***A1***

($\Rightarrow$ coordinates are $(\mathrm{e}+2, 1)$)

[9 marks]

37. **EITHER**

$\int_1^{\mathrm{e}} \dfrac{(\ln x)^3}{x}\mathrm{d}x$

$y = (\ln x)^4$ ***M2***

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4(\ln x)^3}{x}$ ***A1***

$\int_1^{\mathrm{e}} \dfrac{(\ln x)^3}{x}\mathrm{d}x = \dfrac{1}{4}\left[(\ln x)^4\right]_1^{\mathrm{e}}$ ***A1***

$= \dfrac{1}{4}[1-0] = \dfrac{1}{4}$ ***A1***

OR

Let $u = \ln x$ ***M1***

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$ ***A1***

When $x = 1, u = 0$ and when $x = \mathrm{e}, u = 1$ ***A1***

$\Rightarrow \int_0^1 u^3\,\mathrm{d}u$ ***A1***

$\Rightarrow \left[\dfrac{1}{4}u^4\right]_0^1 = \dfrac{1}{4}$ ***A1***

[5 marks]

38. $4-x^2 \geq 0$ for $0 \leq x \leq 2$ ***A1***

and $4-x^2 \leq 0$ for $2 \leq x \leq 4$ ***A1***

$I = \int_0^2 (4-x^2)\,\mathrm{d}x + \int_2^4 (x^2-4)\,\mathrm{d}x$ ***M1A1***

$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^4$ ***A1A1***

$= 8 - \frac{8}{3} + \frac{64}{3} - 16 - \frac{8}{3} + 8 \quad (=16)$ ***A1***

[7 marks]

39. $x = 2\sin\theta$

$$x^2 = 4\sin^2\theta$$
$$4 - x^2 = 4 - 4\sin^2\theta$$
$$= 4(1-\sin^2\theta)$$
$$= 4\cos^2\theta$$
$$\sqrt{4-x^2} = 2\cos\theta \qquad \textbf{\textit{A1}}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta \qquad \textbf{\textit{M1}}$$

When $x = 1$, $2\sin\theta = 1$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6} \qquad \textbf{\textit{A1}}$$

When $x = \sqrt{3}$, $2\sin\theta = \sqrt{3}$

$$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3} \qquad \textbf{\textit{A1}}$$

Let $I = \int_1^{\sqrt{3}} \sqrt{4-x^2}\,\mathrm{d}x$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\cos\theta \times 2\cos\theta\,\mathrm{d}\theta$$

$$\Rightarrow I = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2\theta\,\mathrm{d}\theta \qquad \textbf{\textit{A1}}$$

Now $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$$\Rightarrow I = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 2\theta + 1\,\mathrm{d}\theta \qquad \textbf{\textit{M1A1}}$$

$$\Rightarrow I = 2\left(\frac{1}{2}\sin 2\theta + \theta\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \qquad \textbf{\textit{M1A1}}$$

$$\Rightarrow I = 2\left(\frac{1}{2}\sin\frac{2\pi}{3} + \frac{\pi}{3}\right) - 2\left(\frac{1}{2}\sin\frac{\pi}{3} + \frac{\pi}{6}\right) \qquad \textbf{\textit{(M1)}}$$

$$\Rightarrow I = \frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{2} - \frac{\pi}{3} \quad \left(= \frac{\pi}{3}\right) \qquad \textbf{\textit{A1}}$$

[11 marks]

40. (a) $\text{OP} = \sqrt{a^2 + (a^2 - 5)^2}$ ***M1***

$= \sqrt{a^2 + a^4 - 10a^2 + 25}$ ***A1***

$= \sqrt{a^4 - 9a^2 + 25}$ ***AG***

[2 marks]

(b) **EITHER**

Let $s = \sqrt{a^4 - 9a^2 + 25}$

$\Rightarrow s^2 = a^4 - 9a^2 + 25$

$\frac{\mathrm{d}s^2}{\mathrm{d}a} = 4a^3 - 18a = 0$ ***M1A1***

$\frac{\mathrm{d}s^2}{\mathrm{d}a} = 0$ for minimum ***(M1)***

$\Rightarrow 2a(2a^2 - 9) = 0$

$\Rightarrow a^2 = \frac{9}{2}$

$\Rightarrow a = \pm\frac{3}{\sqrt{2}} \quad \left(= \pm\frac{3\sqrt{2}}{2}\right)$ ***A1A1***

OR

$s = (a^4 - 9a^2 + 25)^{\frac{1}{2}}$

$\frac{\mathrm{d}s}{\mathrm{d}a} = \frac{1}{2}(a^4 - 9a^2 + 25)^{-\frac{1}{2}}(4a^3 - 18a)$ ***M1A1***

$\frac{\mathrm{d}s}{\mathrm{d}a} = 0$ for a minimum ***(M1)***

$4a^3 - 18a = 0$

$\Rightarrow 2a(2a^2 - 9) = 0$

$\Rightarrow a^2 = \frac{9}{2}$

$\Rightarrow a = \pm\frac{3}{\sqrt{2}} \quad \left(= \pm\frac{3\sqrt{2}}{2}\right)$ ***A1A1***

[5 marks]

Total [7 marks]

41. **EITHER**

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \sin x(\cos x)^{-\frac{1}{2}}\,\mathrm{d}x$$ ***(M1)***

$$= \left[-\frac{\cos^{\frac{1}{2}} x}{\frac{1}{2}}\right]_0^{\frac{\pi}{4}}$$ ***(M1)A1A1***

$$= \left[-2\sqrt{\cos x}\right]_0^{\frac{\pi}{4}}$$

$$= -2\sqrt{\cos\frac{\pi}{4}} + 2\sqrt{\cos 0}$$ ***A1A1***

$$= 2 - 2^{\frac{3}{4}}$$ ***A1***

OR

Let $u = \cos x$ ***(M1)***

$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$ ***(M1)***

when $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$ ***A1***

when $x = 0$, $u = 1$ ***A1***

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}}\,\mathrm{d}x = \int_1^{\frac{1}{\sqrt{2}}} -\frac{1}{u^{\frac{1}{2}}}\,\mathrm{d}u = \int_1^{\frac{1}{\sqrt{2}}} -u^{-\frac{1}{2}}\,\mathrm{d}u$$ ***(M1)***

$$= \left[-2u^{\frac{1}{2}}\right]_1^{\frac{1}{\sqrt{2}}}$$ ***A1***

$$= -\frac{2}{2^{\frac{1}{4}}} + 2 \quad \left(= 2 - 2^{\frac{3}{4}}\right)$$ ***A1***

[7 marks]

42. Substituting $u = x + 2 \Rightarrow u - 2 = x$, $\mathrm{d}u = \mathrm{d}x$ ***(M1)***

$$\int \frac{x^3}{(x+2)^2}\,\mathrm{d}x = \int \frac{(u-2)^3}{u^2}\,\mathrm{d}u$$ ***A1***

$$= \int \frac{u^3 - 6u^2 + 12u - 8}{u^2}\,\mathrm{d}u$$ ***A1***

$$= \int u\,\mathrm{d}u + \int (-6)\,\mathrm{d}u + \int \frac{12}{u}\,\mathrm{d}u - \int 8u^{-2}\,\mathrm{d}u$$ ***A1***

$$= \frac{u^2}{2} - 6u + 12\ln|u| + 8u^{-1} + c$$ ***A1***

$$= \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c$$ ***A1*** ***N0***

[6 marks]

Section B

43. (a) Let $p = 2$, $\Rightarrow 8 + 4 - 10 - 2 = 0$ ***M1***

Since this fits $p = 2$ is a solution. ***R1***

[2 marks]

(b) $p^3 + p^2 - 5p - 2 = (p-2)(p^2 + ap + b)$

$= p^3 + ap^2 + bp - 2p^2 - 2ap - 2b$ ***M1A1***

$= p^3 + p^2(a-2) + p(b-2a) - 2b$

Equate constants $\Rightarrow -2 = -2b$

$b = 1$ ***A1***

Equate coefficients of p^2 $a - 2 = 1$

$\Rightarrow a = 3$ ***A1***

[4 marks]

(c) $p^2 + 3p + 1 = 0$ ***M1***

$$p = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$ ***A1A1***

[3 marks]

(d) (i) Arithmetic sequence: $1,\ 1+p,\ 1+2p,\ 1+3p$ ***A1***

Geometric sequence: $1,\ p,\ p^2,\ p^3$ ***A1***

(ii) $(1+2p) + (1+3p) = p^2 + p^3$ ***M1A1***

$\Rightarrow p^3 + p^2 - 5p - 2 = 0$ ***A1***

Therefore, from part (i), $p = 2$, $p = \dfrac{-3 \pm \sqrt{5}}{2}$ ***R1***

(iii) The sum to infinity of a geometric series exists if $|p| < 1$. ***R1***

Hence, $p = \dfrac{-3 + \sqrt{5}}{2}$ is the only such number. ***A1***

(iv) The sum of the first 20 terms of the arithmetic series can be found by applying the sum formula

$S_{20} = 10(2a + 19d) = 10(2 + 19p)$ ***M1A1***

So, $S_{20} = 10\left(2 + 19\left(\dfrac{\sqrt{5} - 3}{2}\right)\right) = -265 + 95\sqrt{5}$ ***A1A1A1***

[13 marks]

Total [22 marks]

44. **Part A**

Let $f(n) = 5^n + 9^n + 2$ and let P_n be the proposition that $f(n)$ is divisible by 4.

Then $f(1) = 16$ ***A1***

So P_1 is true ***A1***

Let P_n be true for $n = k$ *i.e.* $f(k)$ is divisible by 4 ***M1***

Consider $f(k+1) = 5^{k+1} + 9^{k+1} + 2$ ***M1***

$= 5^k(4+1) + 9^k(8+1) + 2$ ***A1***

$= f(k) + 4(5^k + 2 \times 9^k)$ ***A1***

Both terms are divisible by 4 so $f(k+1)$ is divisible by 4. ***R1***

P_k true $\Rightarrow P_{k+1}$ true ***R1***

Since P_1 is true, P_n is proved true by mathematical induction for $n \in \mathbb{Z}^+$. ***R1*** ***N0***

[9 marks]

Part B

(a) $z = \dfrac{\frac{1}{2}e^{2i\theta}}{e^{i\theta}}$ ***(M1)***

$z = \frac{1}{2}e^{i\theta}$ ***A1*** ***N2***

[2 marks]

(b) $|z| = \frac{1}{2}$ ***A2***

$|z| < 1$ ***AG***

[2 marks]

(c) Using $S_\infty = \dfrac{a}{1-r}$ ***(M1)***

$S_\infty = \dfrac{e^{i\theta}}{1 - \frac{1}{2}e^{i\theta}}$ ***A1*** ***N2***

[2 marks]

continued ...

Question 44 Part B continued

(d) (i) $S_\infty = \dfrac{e^{i\theta}}{1-\frac{1}{2}e^{i\theta}} = \dfrac{\text{cis}\,\theta}{1-\frac{1}{2}\text{cis}\,\theta}$ ***(M1)***

$\dfrac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}$ ***(A1)***

Also $S_\infty = e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$

$= \text{cis}\,\theta + \frac{1}{2}\text{cis}\,2\theta + \frac{1}{4}\text{cis}\,3\theta + \ldots$ ***(M1)***

$S_\infty = \left(\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots\right) + i\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + \ldots\right)$ ***A1***

(ii) Taking real parts,

$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \text{Re}\left(\dfrac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}\right)$ ***A1***

$= \text{Re}\left(\dfrac{(\cos\theta + i\sin\theta)}{\left(1-\frac{1}{2}\cos\theta - \frac{1}{2}i\sin\theta\right)} \times \dfrac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{\left(1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta\right)}\right)$ ***M1***

$= \dfrac{\cos\theta - \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2 + \frac{1}{4}\sin^2\theta}$ ***A1***

$= \dfrac{\left(\cos\theta - \frac{1}{2}\right)}{1-\cos\theta + \frac{1}{4}(\sin^2\theta + \cos^2\theta)}$ ***A1***

$= \dfrac{(2\cos\theta - 1)\div 2}{(4-4\cos\theta+1)\div 4} = \dfrac{4(2\cos\theta-1)}{2(5-4\cos\theta)}$ ***A1***

$= \dfrac{4\cos\theta - 2}{5-4\cos\theta}$ ***A1AG*** ***N0***

[10 marks]

Total [25 marks]

45. (a) $z = \dfrac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \mathrm{i}\sqrt{3}$ ***M1***

$-1 + \mathrm{i}\sqrt{3} = r\mathrm{e}^{\mathrm{i}\theta} \quad \Rightarrow \quad r = 2$ ***A1***

$\theta = \arctan \dfrac{\sqrt{3}}{-1} = \dfrac{2\pi}{3}$ ***A1***

$-1 - \mathrm{i}\sqrt{3} = r\mathrm{e}^{\mathrm{i}\theta} \quad \Rightarrow \quad r = 2$

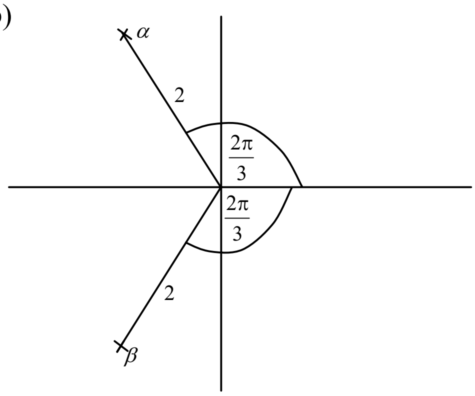
$\theta = \arctan \dfrac{\sqrt{3}}{-1} = -\dfrac{2\pi}{3}$ ***A1***

$\Rightarrow \quad \alpha = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ ***A1***

$\Rightarrow \quad \beta = 2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$ ***A1***

[6 marks]

(b)



A1A1

[2 marks]

(c) $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$

Let $n = 1$

Left hand side $= \cos 1\theta + \mathrm{i}\sin 1\theta = \cos\theta + \mathrm{i}\sin\theta$

Right hand side $= (\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta$

Hence true for $n = 1$ ***M1A1***

Assume true for $n = k$ ***M1***

$\cos k\theta + \mathrm{i}\sin k\theta = (\cos\theta + \mathrm{i}\sin\theta)^k$

$\Rightarrow \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta = (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ ***M1A1***

$= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + \mathrm{i}(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$ ***A1***

$= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$ ***A1***

Hence if true for $n = k$, true for $n = k + 1$

However if it is true for $n = 1$

$\Rightarrow$ true for $n = 2$ *etc.* ***R1***

$\Rightarrow$ hence proved by induction

[8 marks]

continued ...

Question 45 continued

(d) $\frac{\alpha^3}{\beta^2} = \frac{8e^{i2\pi}}{4e^{-i\frac{4\pi}{3}}} = 2e^{i\frac{4\pi}{3}}$ ***A1***

$= 2\cos\frac{4\pi}{3} + 2i\sin\frac{4\pi}{3}$ ***(M1)***

$= -\frac{2}{2} - 2\frac{i\sqrt{3}}{2} = -1 - i\sqrt{3}$ ***A1A1***

[4 marks]

(e) $\alpha^3 = 8e^{i2\pi}$ ***A1***

$\beta^3 = 8e^{-i2\pi}$ ***A1***

Since $e^{2\pi}$ and $e^{-2\pi}$ are the same $\alpha^3 = \beta^3$ ***R1***

[3 marks]

(f) **EITHER**

$\alpha = -1 + i\sqrt{3} \qquad \beta = -1 - i\sqrt{3}$

$\alpha^* = -1 - i\sqrt{3} \qquad \beta^* = -1 + i\sqrt{3}$ ***A1***

$\alpha\beta^* = \left(-1 + i\sqrt{3}\right)\left(-1 + i\sqrt{3}\right) = 1 - 2i\sqrt{3} - 3 = 2 - 2i\sqrt{3}$ ***M1A1***

$\beta\alpha^* = \left(-1 - i\sqrt{3}\right)\left(-1 - i\sqrt{3}\right) = 1 + 2i\sqrt{3} - 3 = -2 + 2i\sqrt{3}$ ***A1***

$\Rightarrow \alpha\beta^* + \beta\alpha^* = -4$ ***A1***

OR

Since $\alpha^* = \beta$ and $\beta^* = \alpha$

$\alpha\beta^* = 2e^{i\frac{2\pi}{3}} \times 2e^{i\frac{2\pi}{3}} = 4e^{i\frac{4\pi}{3}}$ ***M1A1***

$\beta\alpha^* = 2e^{-i\frac{2\pi}{3}} \times 2e^{-i\frac{2\pi}{3}} = 4e^{-i\frac{4\pi}{3}}$ ***A1***

$\alpha\beta^* + \beta\alpha^* = 4\left(e^{i\frac{4\pi}{3}} + e^{-i\frac{4\pi}{3}}\right)$

$= 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} + \cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$ ***A1***

$= 8\cos\frac{4\pi}{3} = 8 \times -\frac{1}{2} = -4$ ***A1***

[5 marks]

(g) $\alpha^n = 2^n e^{i2\frac{\pi n}{3}}$ ***M1A1***

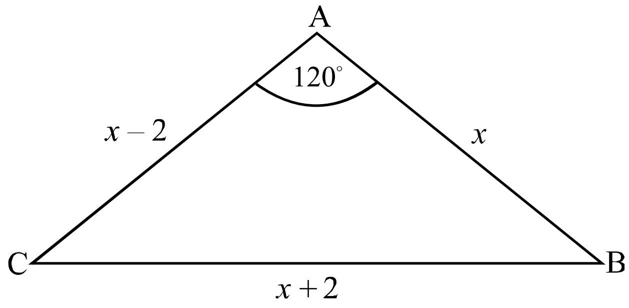
This is real when n is a multiple of 3 ***R1***

i.e. $n = 3N$ where $N \in \mathbb{Z}^+$

[3 marks]

Total [31 marks]

46. (a)



(M1)

$(x+2)^2 = (x-2)^2 + x^2 - 2(x-2)x\cos 120^\circ$ ***M1A1***

$x^2 + 4x + 4 = x^2 - 4x + 4 + x^2 + x^2 - 2x$ ***(M1)***

$0 = 2x^2 - 10x$ ***A1***

$0 = x(x-5)$

$x = 5$ ***A1***

[6 marks]

(b) Area $= \frac{1}{2} \times 5 \times 3 \times \sin 120^\circ$ ***M1A1***

$= \frac{1}{2} \times 15 \times \frac{\sqrt{3}}{2}$ ***A1***

$= \frac{15\sqrt{3}}{4}$ ***AG***

[3 marks]

(c) $\sin A = \frac{\sqrt{3}}{2}$

$\frac{15\sqrt{3}}{4} = \frac{1}{2} \times 5 \times 7 \times \sin B \Rightarrow \sin B = \frac{3\sqrt{3}}{14}$ ***M1A1***

Similarly $\sin C = \frac{5\sqrt{3}}{14}$ ***A1***

$\sin A + \sin B + \sin C = \frac{15\sqrt{3}}{14}$ ***A1***

[4 marks]

Total [13 marks]

47. (a) $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 3 & 5 & -1 & 0 \\ 1 & -5 & 2-a & 9-a^2 \end{array}\right]$ ***M1***

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & -8 & 3-a & 9-a^2 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$ ***(M1)***

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -a-1 & 9-a^2 \end{array}\right] \begin{array}{l} R_2 \to R_2 \times -\frac{1}{2} \\ R_3 \to R_3 - 2R_2 \end{array}$ ***M1***

When $a = -1$ the augmented matrix is

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 8 \end{array}\right]$ ***A1***

Hence the system is inconsistent $\Rightarrow\ a \neq -1$ ***R1***

[5 marks]

(b) When $a \neq -1$, $(-a-1)z = 9 - a^2$

$(a+1)z = a^2 - 9$

$\therefore z = \dfrac{a^2-9}{a+1}$ ***M1A1***

$2y - z = 0 \Rightarrow y = \dfrac{1}{2}z = \dfrac{a^2-9}{2(a+1)}$ ***M1A1***

$x = -3y + z = \dfrac{-3(a^2-9)}{2(a+1)} + \dfrac{2(a^2-9)}{2(a+1)} = \dfrac{9-a^2}{2(a+1)}$ ***M1A1***

The unique solution is $\left(\dfrac{9-a^2}{2(a+1)}, \dfrac{a^2-9}{2(a+1)}, \dfrac{a^2-9}{a+1}\right)$ when $a \neq -1$

[6 marks]

(c) $2 - a = 1 \quad \Rightarrow \quad a = 1$ ***M1***

$\therefore$ The solution is $\left(\dfrac{8}{4}, -\dfrac{8}{4}, -\dfrac{8}{2}\right)$ or $(2, -2, -4)$ ***A1***

[2 marks]

Total [13 marks]

48. (a) $\vec{AB} = -\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\vec{BC} = \boldsymbol{i} + \boldsymbol{j}$ *A1A1*

[2 marks]

(b) $\vec{AB} \times \vec{BC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ *M1*

$= -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ *A1*

[2 marks]

(c) Area of Δ ABC $= \frac{1}{2}\left| -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k} \right|$ *M1A1*

$= \frac{1}{2}\sqrt{1+1+4}$

$= \frac{\sqrt{6}}{2}$ *A1*

[3 marks]

(d) A normal to the plane is given by $\boldsymbol{n} = \vec{AB} \times \vec{BC} = -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ *(M1)*
Therefore, the equation of the plane is of the form $-x + y + 2z = g$
and since the plane contains A, then $-1 + 2 + 2 = g \Rightarrow g = 3$. *M1*
Hence, an equation of the plane is $-x + y + 2z = 3$. *A1*

[3 marks]

(e) Vector $\boldsymbol{n}$ above is parallel to the required line.

Therefore, $x = 2 - t$ *A1*
$y = -1 + t$ *A1*
$z = -6 + 2t$ *A1*

[3 marks]

(f) $x = 2 - t$
$y = -1 + t$
$z = -6 + 2t$

$-x + y + 2z = 3$
$-2 + t - 1 + t - 12 + 4t = 3$ *M1A1*
$-15 + 6t = 3$
$6t = 18$
$t = 3$ *A1*
Point of intersection $(-1, 2, 0)$ *A1*

[4 marks]

(g) Distance $= \sqrt{3^2 + 3^2 + 6^2} = \sqrt{54}$ *(M1)A1*

[2 marks]

continued ...

Question 48 continued

(h) Unit vector in the direction of $\boldsymbol{n}$ is $e = \frac{1}{|\boldsymbol{n}|} \times \boldsymbol{n}$ ***(M1)***

$$= \frac{1}{\sqrt{6}}(-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$$ ***A1***

Note: $-e$ is also acceptable.

[2 marks]

(i) Point of intersection of L and P is $(-1, 2, 0)$.

$\vec{DE} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$ ***(M1)A1***

$\Rightarrow \vec{EF} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$ ***M1***

$\Rightarrow$ coordinates of F are $(-4, 5, 6)$ ***A1***

[4 marks]

Total [25 marks]

49. (a) $L_1 : x = 2 + \lambda \,;\; y = 2 + 3\lambda \,;\; z = 3 + \lambda$ ***(A1)***

$L_2 : x = 2 + \mu \,;\; y = 3 + 4\mu \,;\; z = 4 + 2\mu$ ***(A1)***

At the point of intersection ***(M1)***

$2 + \lambda = 2 + \mu \quad (1)$

$2 + 3\lambda = 3 + 4\mu \quad (2)$

$3 + \lambda = 4 + 2\mu \quad (3)$

From (1), $\lambda = \mu$ ***A1***

Substituting in (2), $2 + 3\lambda = 3 + 4\lambda$

$\Rightarrow \lambda = \mu = -1$ ***A1***

We need to show that these values satisfy (3). ***(M1)***

They do because LHS = RHS = 2; therefore the lines intersect. ***R1***

So P is $(1, -1, 2)$. ***A1*** ***N3***

[8 marks]

(b) The normal to Π is normal to both lines. It is therefore given by the vector product of the two direction vectors.

Therefore, normal vector is given by $\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{pmatrix}$ ***M1A1***

$= 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ ***A2***

The Cartesian equation of Π is $2x - y + z = 2 + 1 + 2$ ***(M1)***

i.e. $2x - y + z = 5$ ***A1*** ***N2***

[6 marks]

(c) The midpoint M of [PQ] is $\left(2, \frac{3}{2}, \frac{5}{2}\right)$. ***M1A1***

The direction of $\overrightarrow{\text{MS}}$ is the same as the normal to Π, *i.e.* $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ ***(R1)***

The coordinates of a general point R on $\overrightarrow{\text{MS}}$ are therefore

$\left(2 + 2\lambda, \frac{3}{2} - \lambda, \frac{5}{2} + \lambda\right)$ ***(M1)***

It follows that $\overrightarrow{\text{PR}} = (1 + 2\lambda)\boldsymbol{i} + \left(\frac{5}{2} - \lambda\right)\boldsymbol{j} + \left(\frac{1}{2} + \lambda\right)\boldsymbol{k}$ ***A1A1A1***

At S, length of $\overrightarrow{\text{PR}}$ is 3, *i.e.* ***(M1)***

$(1 + 2\lambda)^2 + \left(\frac{5}{2} - \lambda\right)^2 + \left(\frac{1}{2} + \lambda\right)^2 = 9$ ***A1***

$1 + 4\lambda + 4\lambda^2 + \frac{25}{4} - 5\lambda + \lambda^2 + \frac{1}{4} + \lambda + \lambda^2 = 9$ ***(A1)***

$6\lambda^2 = \frac{6}{4}$ ***A1***

$\lambda = \pm\frac{1}{2}$ ***A1***

Substituting these values, ***(M1)***

the possible positions of S are $(3, 1, 3)$ and $(1, 2, 2)$ ***A1A1*** ***N2***

[15 marks]

Total [29 marks]

50. (a) Using $\int_0^1 f(x)\,dx = 1$ ***(M1)***

$k\int_0^1 \frac{dx}{\sqrt{4-x^2}} = 1$ ***A1***

$k\left[\arcsin\frac{x}{2}\right]_0^1 = 1$ ***A1***

$k\left(\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right) = 1$ ***A1***

$k \times \frac{\pi}{6} = 1$

$k = \frac{6}{\pi}$ ***A1***

[5 marks]

(b) $E(X) = \frac{6}{\pi}\int_0^1 \frac{x\,dx}{\sqrt{4-x^2}}$ ***M1***

Let $u = 4 - x^2$ ***(M1)***

$\frac{du}{dx} = -2x$ ***A1***

When $x = 0$, $u = 4$ ***A1***

When $x = 1$, $u = 3$ ***A1***

$E(X) = -\frac{6}{\pi} \times \frac{1}{2}\int_4^3 \frac{du}{u^{\frac{1}{2}}}$ ***M1***

$= -\frac{6}{\pi}\left[u^{\frac{1}{2}}\right]_4^3$ ***A1***

$= \frac{6}{\pi}(2 - \sqrt{3})$ ***AG***

[7 marks]

continued ...

Question 50 continued

(c) The median m satisfies

$$\frac{6}{\pi}\int_0^m \frac{\mathrm{d}x}{\sqrt{4-x^2}} = \frac{1}{2}$$ ***M1A1***

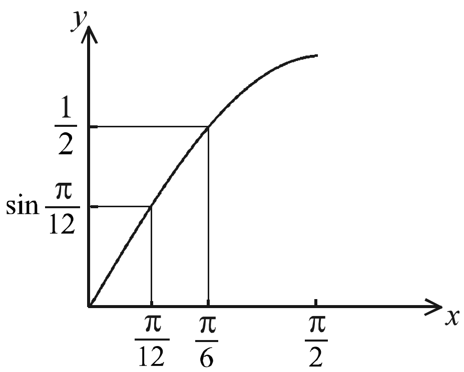
$$\frac{6}{\pi}\left[\arcsin\left(\frac{x}{2}\right)\right]_0^m = \frac{1}{2}$$ ***A1***

$$\arcsin\left(\frac{m}{2}\right) = \frac{\pi}{12}$$ ***A1***

$$m = 2\sin\left(\frac{\pi}{12}\right)$$ ***A1***

We need to determine whether $2\sin\frac{\pi}{12} >$ or $< \frac{1}{2}$

Consider the graph of $y = \sin x$ ***M1***



Since the graph of $y = \sin x$ for $0 \le x \le \frac{\pi}{2}$ is concave downwards and $\sin\frac{\pi}{6} = \frac{1}{2}$

it follows by inspection that $\sin\frac{\pi}{12} > \frac{1}{4}$ ***R1***

hence $m = 2\sin\frac{\pi}{12} > \frac{1}{2}$ ***R1***

[8 marks]

Total [20 marks]

51. (a) $P(RR) = \left(\frac{2}{5}\right)\left(\frac{1}{4}\right)$ ***(M1)***

$= \frac{1}{10}$ ***A1*** ***N2***

[2 marks]

(b) $P(RR) = \frac{4}{4+n} \times \frac{3}{3+n} = \frac{2}{15}$ ***A1***

Forming equation $12 \times 15 = 2(4+n)(3+n)$ ***(M1)***

$12 + 7n + n^2 = 90$ ***A1***

$\Rightarrow n^2 + 7n - 78 = 0$ ***A1***

$n = 6$ ***AG*** ***N0***

[4 marks]

(c) **EITHER**

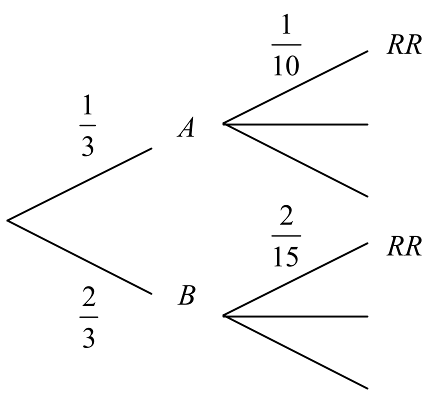
$P(A) = \frac{1}{3}$ $P(B) = \frac{2}{3}$ ***A1***

$P(RR) = P(A \cap RR) + P(B \cap RR)$ ***(M1)***

$= \left(\frac{1}{3}\right)\left(\frac{1}{10}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{15}\right)$

$= \frac{11}{90}$ ***A1*** ***N2***

OR



A1

$P(RR) = \frac{1}{3} \times \frac{1}{10} + \frac{2}{3} \times \frac{2}{15}$ ***M1***

$= \frac{11}{90}$ ***A1*** ***N2***

[3 marks]

continued …

Question 51 continued

(d) $\mathrm{P}(1 \text{ or } 6) = \mathrm{P}(A)$ ***M1***

$\mathrm{P}\left(A \middle| RR\right) = \dfrac{\mathrm{P}(A \cap RR)}{\mathrm{P}(RR)}$ ***(M1)***

$= \dfrac{\left[\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)\right]}{\frac{11}{90}}$ ***M1***

$= \dfrac{3}{11}$ ***A1*** ***N2***

[4 marks]

Total [13 marks]

52. (a) (i) $18(x-1)=0 \Rightarrow x=1$ ***A1***

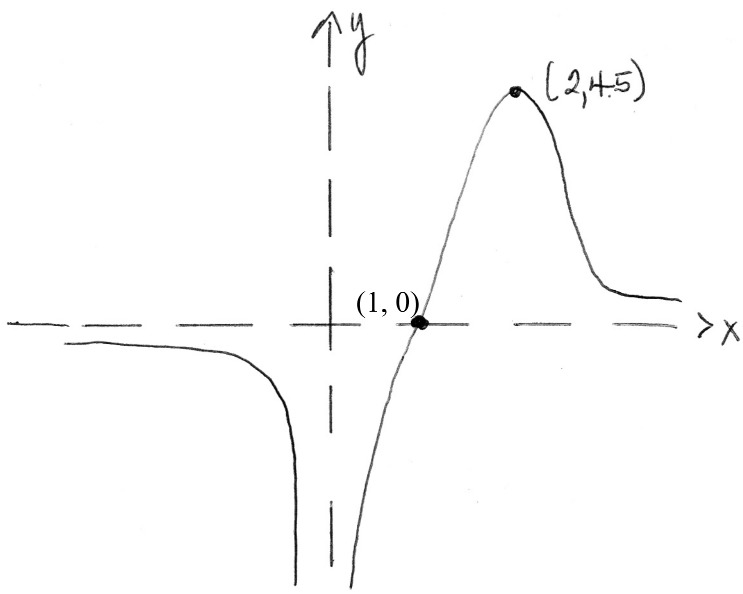
(ii) vertical asymptote: $x=0$ ***A1***
horizontal asymptote: $y=0$ ***A1***

(iii) $18(2-x)=0 \Rightarrow x=2$ ***M1A1***
$f''(2)=\frac{36(2-3)}{2^3}=-\frac{9}{2}<0$ hence it is a maximum point ***R1***
When $x=2$, $f(x)=\frac{9}{2}$ ***A1***
$\left(f(x) \text{ has a maximum at } \left(2, \frac{9}{2}\right)\right)$

(iv) $f(x)$ is concave up when $f''(x)>0$ ***M1***
$36(x-3)>0 \Rightarrow x>3$ ***A1***

[9 marks]

(b)



A1A1A1A1A1

Note:	Award ***A1*** for shape, ***A1*** for maximum, ***A1*** for x-intercept, ***A1*** for horizontal asymptote and ***A1*** for vertical asymptote.

[5 marks]

Total [14 marks]

53. (a) (i) Attempting to use quotient rule $f'(x) = \dfrac{x\dfrac{1}{x} - \ln x \times 1}{x^2}$ ***(M1)***

$f'(x) = \dfrac{1 - \ln x}{x^2}$ ***A1***

$f''(x) = \dfrac{x^2\left(-\dfrac{1}{x}\right) - (1 - \ln x)2x}{x^4}$ ***(M1)***

$f''(x) = \dfrac{2\ln x - 3}{x^3}$ ***A1***

Stationary point where $f'(x) = 0$, ***M1***

i.e. $\ln x = 1$, (so $x = \mathrm{e}$) ***A1***

$f''(\mathrm{e}) < 0$ so maximum. ***R1AG*** ***N0***

(ii) Exact coordinates $x = \mathrm{e}$, $y = \dfrac{1}{\mathrm{e}}$ ***A1A1*** ***N2***

(iii) Solving $f''(0) = 0$ ***M1***

$\ln x = \dfrac{3}{2}$ ***(A1)***

$x = \mathrm{e}^{\frac{3}{2}}$ ***A1*** ***N2***

[12 marks]

continued ...

Question 53 continued

(b) Area $= \int_1^5 \frac{\ln x}{x} \mathrm{d}x$ ***A1***

EITHER

Finding the integral by substitution/inspection

$u = \ln x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ ***(M1)***

$\int u \,\mathrm{d}u = \frac{u^2}{2}\left(= \frac{(\ln x)^2}{2}\right)$ ***M1A1***

Area $= \left[\frac{(\ln x)^2}{2}\right]_1^5 = \frac{1}{2}\left((\ln 5)^2 - (\ln 1)^2\right)$ ***A1***

Area $= \frac{1}{2}(\ln 5)^2$ ***A1*** ***N2***

OR

Finding the integral I by parts ***(M1)***

$u = \ln x$, $\mathrm{d}v = \frac{1}{x} \Rightarrow \mathrm{d}u = \frac{1}{x}$, $v = \ln x$

$I = uv - \int u \,\mathrm{d}v = (\ln x)^2 - \int \ln x \frac{1}{x} \mathrm{d}x = (\ln x)^2 - I$ ***M1***

$\Rightarrow 2I = (\ln x)^2 \Rightarrow I = \frac{(\ln x)^2}{2}$ ***A1***

$\Rightarrow$ Area $= \left[\frac{(\ln x)^2}{2}\right]_1^5 = \frac{1}{2}\left((\ln 5)^2 - (\ln 1)^2\right)$ ***A1***

Area $= \frac{1}{2}(\ln 5)^2$ ***A1*** ***N2***

[6 marks]

Total [18 marks]

54. (a) $\ln e^{2-2x} = \ln 2e^{-x}$ ***M1***

$2-2x = \ln(2e^{-x})$ ***(A1)***

$= \ln 2 - x$ ***(A1)***

$x = 2 - \ln 2$ ***A1***

$\left(x = \ln e^2 - \ln 2 = \ln \dfrac{e^2}{2} \right)$

[4 marks]

(b) $\dfrac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$ ***M1A1***

$\dfrac{dy}{dx} = 0$ for a minimum point ***(M1)***

$-2e^{2-2x} + 2e^{-x} = 0$

$\Rightarrow e^{2-2x} = e^{-x}$ ***(A1)***

$\Rightarrow 2-2x = -x$ ***(A1)***

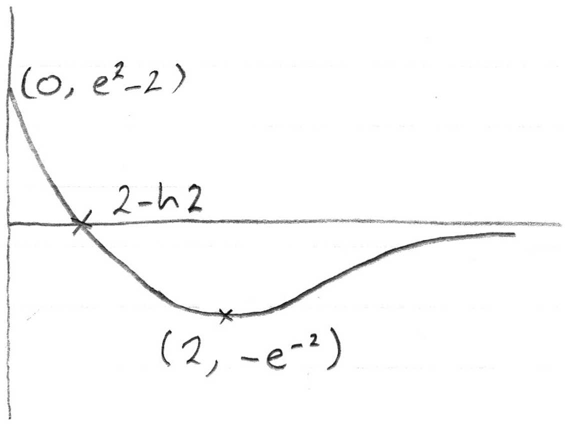
$\Rightarrow x = 2$ ***A1***

$\Rightarrow y = e^{-2} - 2e^{-2} = -e^{-2}$ ***A1***

($\Rightarrow$ minimum point is $(2, -e^{-2})$)

[7 marks]

(c)



A1A1A1

[3 marks]

(d) 2 distinct roots provided $-e^{-2} < k < 0$ ***A1A1***

[2 marks]

Total [16 marks]

55. (a) $f'(x) = \dfrac{2xe^x - x^2e^x}{e^{2x}} \quad \left(= \dfrac{2x - x^2}{e^x}\right)$ ***M1A1***

For a maximum $f'(x) = 0$ ***(M1)***

$2x - x^2 = 0$

giving $x = 0$ or 2 ***A1A1***

$f''(x) = \dfrac{(2-2x)e^x - e^x(2x - x^2)}{e^{2x}} \quad \left(= \dfrac{x^2 - 4x + 2}{e^x}\right)$ ***M1A1***

$f''(0) = 2 > 0 \Rightarrow$ minimum ***R1***

$f''(2) = -\dfrac{2}{e^2} < 0 \Rightarrow$ maximum ***R1***

Maximum value $= \dfrac{4}{e^2}$ ***A1***

[10 marks]

(b) For a point of inflexion,

$f''(x) = \dfrac{x^2 - 4x + 2}{e^x} = 0$ ***M1***

giving $x = \dfrac{4 \pm \sqrt{16 - 8}}{2}$ ***(A1)***

$= 2 \pm \sqrt{2}$ ***A1***

[3 marks]

(c) $\int_0^1 x^2 e^{-x}\,dx = \left[-x^2 e^{-x}\right]_0^1 + 2\int_0^1 x e^{-x}\,dx$ ***M1A1***

$= -e^{-1} - 2\left[x e^{-x}\right]_0^1 + 2\int_0^1 e^{-x}\,dx$ ***A1M1A1***

$= -e^{-1} - 2e^{-1} - 2\left[e^{-x}\right]_0^1$ ***A1A1***

$= -3e^{-1} - 2e^{-1} + 2 \quad \left(= 2 - 5e^{-1}\right)$ ***A1***

[8 marks]

Total [21 marks]

Paper 2

Section A questions

1. *[Maximum mark: 6]*

A sum of $ 5000 is invested at a compound interest rate of 6.3 % per annum.

(a) Write down an expression for the value of the investment after *n* full years.

(b) What will be the value of the investment at the end of five years? *[1 mark]*

(c) The value of the investment will exceed $ 10 000 after *n* full years. *[1 mark]*

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of *n*. *[4 marks]*

2. *[Maximum mark: 6]*

Let $f(x) = \dfrac{x+4}{x+1}$, $x \neq -1$ and $g(x) = \dfrac{x-2}{x-4}$, $x \neq 4$. Find the set of values of x such that $f(x) \leq g(x)$.

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3. *[Maximum mark: 6]*

(a) Write down the inverse of the matrix

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}.$$ *[2 marks]*

(b) **Hence,** find the point of intersection of the three planes.

$$\begin{aligned} x - 3y + z &= 1 \\ 2x + 2y - z &= 2 \\ x - 5y + 3z &= 3 \end{aligned}$$ *[3 marks]*

(c) A fourth plane with equation $x + y + z = d$ passes through the point of intersection. Find the value of d. *[1 mark]*

4. *[Maximum mark: 8]*

A triangle has its vertices at $\mathrm{A}(-1, 3, 2)$, $\mathrm{B}(3, 6, 1)$ and $\mathrm{C}(-4, 4, 3)$.

(a) Show that $\overrightarrow{\mathrm{AB}} \bullet \overrightarrow{\mathrm{AC}} = -10$. *[3 marks]*

(b) Find $\mathrm{B\hat{A}C}$. *[5 marks]*

5. *[Maximum mark: 6]*

The speeds of cars at a certain point on a straight road are normally distributed with mean μ and standard deviation σ. 15 % of the cars travelled at speeds greater than $90\ \mathrm{km\,h^{-1}}$ and 12 % of them at speeds less than $40\ \mathrm{km\,h^{-1}}$. Find μ and σ.

6. *[Maximum mark: 6]*

There are 30 students in a class, of which 18 are girls and 12 are boys. Four students are selected at random to form a committee. Calculate the probability that the committee contains

(a) two girls and two boys; *[3 marks]*

(b) students all of the same gender. *[3 marks]*

7. *[Maximum mark: 6]*

The random variable X has a Poisson distribution with mean 4. Calculate

(a) $\mathrm{P}(3 \le X \le 5)$; *[2 marks]*

(b) $\mathrm{P}(X \ge 3)$; *[2 marks]*

(c) $\mathrm{P}(3 \le X \le 5 | X \ge 3)$. *[2 marks]*

8. *[Maximum mark: 6]*

The displacement s metres of a moving body B from a fixed point O at time t seconds is given by

$$s = 50t - 10t^2 + 1000.$$

(a) Find the velocity of B in $\mathrm{m\,s^{-1}}$. *[2 marks]*

(b) Find its maximum displacement from O. *[4 marks]*

Section B questions

9. *[Maximum mark: 20]*

A farmer owns a triangular field ABC. The side [AC] is 104 m, the side [AB] is 65 m and the angle between these two sides is 60°.

(a) Calculate the length of the third side of the field. *[3 marks]*

(b) Find the area of the field in the form $p\sqrt{3}$, where p is an integer. *[3 marks]*

Let D be a point on [BC] such that [AD] bisects the 60° angle. The farmer divides the field into two parts by constructing a straight fence [AD] of length x metres.

(c) (i) Show that the area of the smaller part is given by $\frac{65x}{4}$ and find an expression for the area of the larger part.

(ii) Hence, find the value of x in the form $q\sqrt{3}$, where q is an integer. *[8 marks]*

(d) Prove that $\frac{\text{BD}}{\text{DC}} = \frac{5}{8}$. *[6 marks]*

10. *[Maximum mark: 12]*

The continuous random variable X has probability density function

$$f(x) = \frac{1}{6}x(1+x^2) \quad \text{for } 0 \le x \le 2,$$
$$f(x) = 0 \quad \text{otherwise.}$$

(a) Sketch the graph of f for $0 \le x \le 2$. *[2 marks]*

(b) Write down the mode of X. *[1 mark]*

(c) Find the mean of X. *[4 marks]*

(d) Find the median of X. *[5 marks]*

Paper 2 markscheme

Section A

1. (a) $5000(1.063)^n$ ***A1*** ***N1***

[1 mark]

(b) Value $= \$\ 5000(1.063)^5$ $(= \$\ 6786.3511\ldots)$

$= \$\ 6790$ to 3 s.f. (accept \$ 6786, or \$ 6786.35) ***A1*** ***N1***

[1 mark]

(c) (i) $5000(1.063)^n > 10\,000$ (or $(1.063)^n > 2$) ***A1*** ***N1***

(ii) Attempting to solve the above inequality $n\log(1.063) > \log 2$ ***(M1)***

$n > 11.345...$ ***(A1)***

12 years ***A1*** ***N3***

Note:	Candidates are likely to use TABLE or LIST on a GDC to find *n*. A good way of communicating this is suggested below.

Let $y = 1.063^x$ ***(M1)***

When $x = 11$, $y = 1.9582$, when $x = 12$, $y = 2.0816$ ***(A1)***

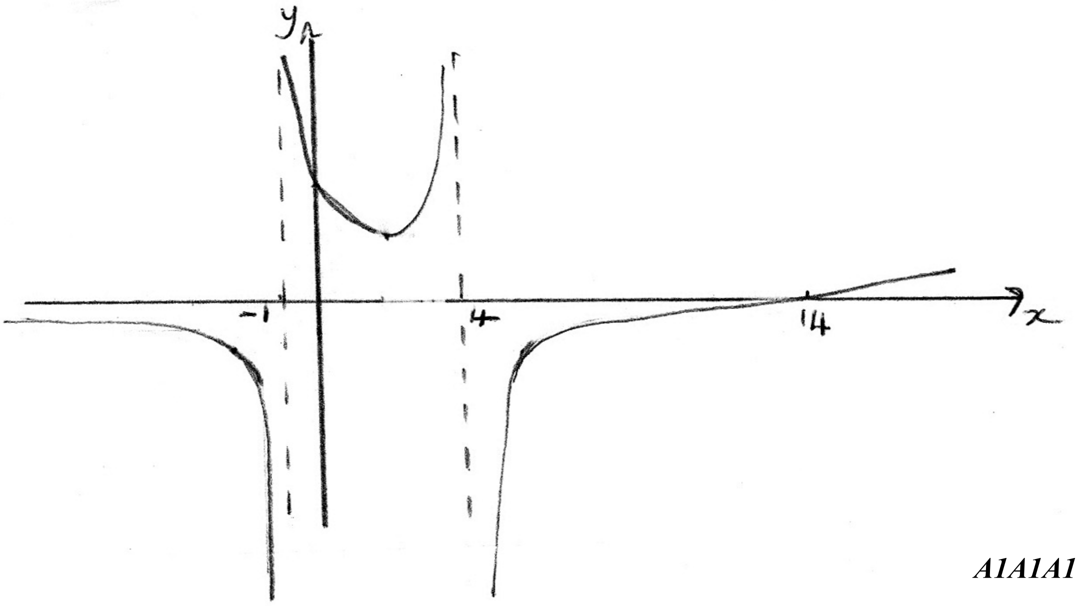
$x = 12$ *i.e.* 12 years ***A1*** ***N3***

[4 marks]

Total [6 marks]

2. **METHOD 1**

Graph of $f(x) - g(x)$ ***M1***



A1A1A1

Note: Award ***A1*** for each branch.

$x < -1$ or $4 < x \leq 14$ ***A1A1*** ***N3***

Note: Each value and inequality sign must be correct.

[6 marks]

METHOD 2

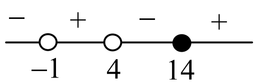
$$\frac{x+4}{x+1} - \frac{x-2}{x-4} \leq 0$$ ***M1***

$$\frac{x^2 - 16 - x^2 + x + 2}{(x+1)(x-4)} \leq 0$$

$$\frac{x-14}{(x+1)(x-4)} \leq 0$$ ***A1***

Critical value of $x = 14$ ***A1***

Other critical values $x = -1$ and $x = 4$ ***A1***



$x < -1$ or $4 < x \leq 14$ ***A1A1*** ***N3***

Note: Each value and inequality sign must be correct.

[6 marks]

3. (a) $A^{-1} = \begin{pmatrix} 0.1 & 0.4 & 0.1 \\ -0.7 & 0.2 & 0.3 \\ -1.2 & 0.2 & 0.8 \end{pmatrix}$ ***A2*** ***N2***

[2 marks]

(b) For attempting to calculate $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ***M1***

$x = 1.2$, $y = 0.6$, $z = 1.6$ (so the point is $(1.2, 0.6, 1.6)$) ***A2*** ***N2***

[3 marks]

(c) $(1.2, 0.6, 1.6)$ lies on $x + y + z = d$

$\therefore d = 3.4$ ***A1*** ***N1***

[1 mark]

Total [6 marks]

4. (a) Finding **correct** vectors $\vec{AB} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ ***A1A1***

Substituting correctly in scalar product $\vec{AB} \cdot \vec{AC} = 4(-3) + 3(1) - 1(1)$ ***A1***

$= -10$ ***AG*** ***N0***

[3 marks]

(b) $\left|\vec{AB}\right| = \sqrt{26}$ $\left|\vec{AC}\right| = \sqrt{11}$ ***(A1)(A1)***

Attempting to use scalar product formula, $\cos B\hat{A}C = \frac{-10}{\sqrt{26}\sqrt{11}}$ ***M1***

$= -0.591$ (to 3 s.f.) ***A1***

$B\hat{A}C = 126^\circ$ ***A1*** ***N3***

[5 marks]

Total [8 marks]

5. $P(X > 90) = 0.15$ **and** $P(X < 40) = 0.12$ ***(M1)***

Finding standardized values 1.036, -1.175 ***A1A1***

Setting up the equations $1.036 = \frac{90 - \mu}{\sigma}$, $-1.175 = \frac{40 - \mu}{\sigma}$ ***(M1)***

$\mu = 66.6$, $\sigma = 22.6$ ***A1A1*** ***N2N2***

[6 marks]

6. (a) Total number of ways of selecting 4 from 30 $=\binom{30}{4}$ *(M1)*

Number of ways of choosing $2B\ 2G=\binom{12}{2}\binom{18}{2}$ *(M1)*

$$\mathrm{P}(2B \text{ or } 2G)=\frac{\binom{12}{2}\binom{18}{2}}{\binom{30}{4}}=0.368$$ *A1* *N2*

[3 marks]

(b) Number of ways of choosing $4B=\binom{12}{4}$, choosing $4G=\binom{18}{4}$ *A1*

$$\mathrm{P}(4B \text{ or } 4G)=\frac{\binom{12}{4}+\binom{18}{4}}{\binom{30}{4}}$$ *(M1)*

$=0.130$ *A1* *N2*

[3 marks]

Total [6 marks]

7. (a) $\mathrm{P}(3\le X\le 5)=\mathrm{P}(X\le 5)-\mathrm{P}(X\le 2)$ *(M1)*

$=0.547$ *A1* *N2*

[2 marks]

(b) $\mathrm{P}(X\ge 3)=1-\mathrm{P}(X\le 2)$ *(M1)*

$=0.762$ *A1* *N2*

[2 marks]

(c) $\mathrm{P}(3\le X\le 5 \mid X\ge 3)=\dfrac{\mathrm{P}(3\le X\le 5)}{\mathrm{P}(X\ge 3)}\quad\left(=\dfrac{0.547}{0.762}\right)$ *(M1)*

$=0.718$ *A1* *N2*

[2 marks]

Total [6 marks]

8. (a) $s = 50t - 10t^2 + 1000$

$v = \frac{\mathrm{d}s}{\mathrm{d}t}$ ***(M1)***

$= 50 - 20t$ ***A1*** ***N2***

[2 marks]

(b) Displacement is max when $v = 0$, ***M1***

i.e. when $t = \frac{5}{2}$. ***A1***

Substituting $t = \frac{5}{2}$, $s = 50 \times \frac{5}{2} - 10 \times \left(\frac{5}{2}\right)^2 + 1000$ ***(M1)***

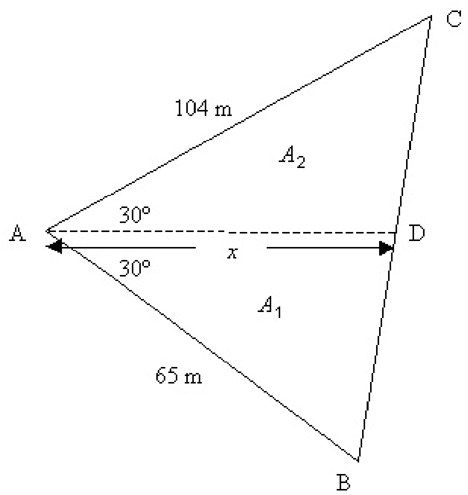
$s = 1062.5$ m ***A1*** ***N2***

[4 marks]

Total [6 marks]

Section B

9.



(a) Using the cosine rule $(a^2 = b^2 + c^2 - 2bc\cos A)$ ***(M1)***

Substituting correctly

$BC^2 = 65^2 + 104^2 - 2(65)(104)\cos 60^\circ$ ***A1***

$= 4225 + 10\,816 - 6760 = 8281$

$\Rightarrow BC = 91\text{ m}$ ***A1*** ***N2***

[3 marks]

(b) Finding the area using $= \frac{1}{2}bc\sin A$ ***(M1)***

Substituting correctly, area $= \frac{1}{2}(65)(104)\sin 60^\circ$ ***A1***

$= 1690\sqrt{3}$ (accept $p = 1690$) ***A1*** ***N2***

[3 marks]

(c) (i) Smaller area $A_1 = \left(\frac{1}{2}\right)(65)(x)\sin 30^\circ$ ***(M1)A1***

$= \frac{65x}{4}$ ***AG*** ***N0***

Larger area $A_2 = \left(\frac{1}{2}\right)(104)(x)\sin 30^\circ$ ***M1***

$= 26x$ ***A1*** ***N1***

(ii) Using $A_1 + A_2 = A$ ***(M1)***

Substituting $\frac{65x}{4} + 26x = 1690\sqrt{3}$ ***A1***

Simplifying $\frac{169x}{4} = 1690\sqrt{3}$ ***A1***

Solving $x = \frac{4 \times 1690\sqrt{3}}{169}$

$\Rightarrow x = 40\sqrt{3}$ (accept $q = 40$) ***A1*** ***N1***

[8 marks]

continued ...

Question 9 continued

(d) Using sin rule in ΔADB and ΔACD ***(M1)***

Substituting correctly $\dfrac{\text{BD}}{\sin 30^\circ} = \dfrac{65}{\sin \text{A}\hat{\text{D}}\text{B}} \Rightarrow \dfrac{\text{BD}}{65} = \dfrac{\sin 30^\circ}{\sin \text{A}\hat{\text{D}}\text{B}}$ ***A1***

and $\dfrac{\text{DC}}{\sin 30^\circ} = \dfrac{104}{\sin \text{A}\hat{\text{D}}\text{C}} \Rightarrow \dfrac{\text{DC}}{104} = \dfrac{\sin 30^\circ}{\sin \text{A}\hat{\text{D}}\text{C}}$ ***A1***

Since $\text{A}\hat{\text{D}}\text{B} + \text{A}\hat{\text{D}}\text{C} = 180^\circ$ ***R1***

It follows that $\sin \text{A}\hat{\text{D}}\text{B} = \sin \text{A}\hat{\text{D}}\text{C}$ ***R1***

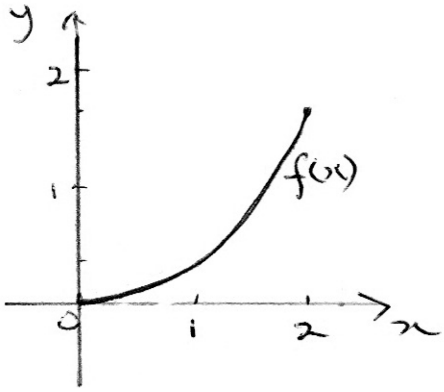
$\dfrac{\text{BD}}{65} = \dfrac{\text{DC}}{104} \Rightarrow \dfrac{\text{BD}}{\text{DC}} = \dfrac{65}{104}$ ***A1***

$\Rightarrow \dfrac{\text{BD}}{\text{DC}} = \dfrac{5}{8}$ ***AG*** ***N0***

[6 marks]

Total [20 marks]

10. (a)



A2

[2 marks]

(b) Mode $= 2$ *A1*

[1 mark]

(c) Using $\mathrm{E}(X) = \int_a^b x\, f(x)\,\mathrm{d}x$ *(M1)*

Mean $= \frac{1}{6}\int_0^2 (x^2 + x^4)\,\mathrm{d}x$ *A1*

$= \frac{1}{6}\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^2$ *(A1)*

$= \frac{68}{45}$ (1.51) *A1* ***N2***

[4 marks]

(d) The median m satisfies $\frac{1}{6}\int_0^m (x + x^3)\,\mathrm{d}x = \frac{1}{2}$ *M1A1*

$\frac{m^2}{2} + \frac{m^4}{4} = 3$ *(A1)*

$\Rightarrow m^4 + 2m^2 - 12 = 0$

$m^2 = \frac{-2 \pm \sqrt{4+48}}{2} = 2.60555...$ *(A1)*

$m = 1.61$ *A1* ***N3***

[5 marks]

Total [12 marks]